Inverse Scattering at a Fixed Quasi–Energy for Potentials Periodic in Time *

arXiv:math-ph/0309043v2 5 Apr 2004

Ricardo Weder†
Instituto de Investigaciones en Matemáticas Aplicadas y en Sistemas.
Universidad Nacional Autónoma de México,
Apartado Postal 20-726, México D.F. 01000. México.
E-Mail: weder@servidor.unam.mx

Abstract

We prove that the scattering matrix at a fixed quasi–energy determines uniquely a time–periodic potential that decays exponentially at infinity. We consider potentials that for each fixed time belong to $L^{3/2}$ in space. The exponent 3/2 is critical for the singularities of the potential in space. For this singular class of potentials the result is new even in the time–independent case, where it was only known for bounded exponentially decreasing potentials.

Short Title: Inverse Scattering at a Fixed Quasi–Energy

1 Introduction

We consider the scattering of a quantum-mechanical particle in $\mathbb{R}^3$ by its interaction with a short–range external potential that is periodic in time. The time–dependent Schrödinger equation is given by,

$$i\frac{\partial}{\partial t}\varphi(t,x) = H(t)\varphi(t,x), \varphi(t_0,x) = \varphi_0(x), \tag{1.1}$$

where $H_0 = -\Delta$, and $H(t) = H_0 + V(t,x)$ are, respectively, the unperturbed and the perturbed Hamiltonians. Assuming that V is real valued and that it satisfies appropriate conditions on its regularity and on its decay as $|x| \to \infty$, that we specify below, and that it is periodic in time, with

*AMS 2000 classification 35P25, 35Q40, 81U40. Research partially supported by project PAPIIT–UNAM, IN 101902.
†Fellow Sistema Nacional de Investigadores.

period that we take as 2π, i.e., $V(t+2\pi,x)=V(t,x)$, the solution to (1.1) is given by a strongly continuous unitary group on L^2,

$$\varphi(t)=U(t,t_0)\varphi_0, \quad \varphi_0 \in L^2, \tag{1.2}$$

where we denote by $\varphi(t)$ the function $\varphi(t,\cdot)$.

The wave operators with time lag $\tau \in \mathbb{R}$ are defined as the following strong limits

$$W_{\pm}(\tau) := \mathrm{s}-\lim_{t\to\pm\infty} U^*(t+\tau,\tau)e^{-itH_0}. \tag{1.3}$$

We give below conditions assuring that the $W_{\pm}(\tau)$ exist and are complete, i.e., Range $W_{\pm}(\tau)=$

$\mathcal{H}_{ac}\left(U(\tau+2\pi,\tau)\right)$. Here $\mathcal{H}_{ac}(U)$ denotes the subspace of absolute continuity of U. Then, the scattering operators

$$S(\tau) := W_+^*(\tau)W_-(\tau), \quad \tau\in\mathbb{R}, \tag{1.4}$$

are unitary on L^2. Note that as V is periodic, $W_{\pm}(\tau+2\pi)=W_{\pm}(\tau)$, and $S(\tau+2\pi)=S(\tau)$. Hence, it is enough to consider $\tau\in[0,2\pi)$. The construction of the scattering matrix associated to $S(\tau)$ is a consequence of the application to our problem of the Howland–Floquet method [9],[43], [10] and of the Kato–Kuroda scattering theory [19], [22]. However, to motivate physically the scattering matrix, and in particular, to clarify what is the meaning, in physical terms, of a scattering experiment at a fixed quasi–energy, it is convenient to briefly discuss how scattering experiments with $S(\tau)$ are related to each other for different $\tau's$. Here we follow [25]. As is well known [29], the wave operators satisfy the following intertwining relations,

$$W_{\pm}(\tau)=U(\tau,0)W_{\pm}(0)e^{i\tau H_0}. \tag{1.5}$$

Hence, the scattering operators satisfy,

$$S(\tau)=e^{-i\tau H_0}S(0)e^{i\tau H_0}. \tag{1.6}$$

The incoming asymptotic states φ and $e^{i\tau H_0}\varphi$ represent two identically prepared states, except for a time lag τ. Then, thinking in terms of the Heisenberg representation of quantum mechanics, the operator $S(\tau)$ describes a scattering experiment corresponding to an incoming asymptotic state

prepared with a time lag τ. Note that particles that enter the interaction region at different times do not interact with the same configuration of the potential. Hence, to consider all possible scattering events we have to take into account the whole family, $S(\tau), \tau \in [0, 2\pi)$. A natural way to do this is to let $S(\tau)$ act as a multiplication operator in the enlarged space,

$$\mathcal{H} := L^2(T, L^2, dt) \tag{1.7}$$

where T is the torus, $T := \mathbb{R}/2\pi\mathbb{Z}$, with $\mathbb{Z}$ the integers, and dt the measure induced in T by Lebesgue measure in $\mathbb{R}$. That is to say, we consider the locally square–integrable functions on $\mathbb{R}$ with values in L^2 that are periodic with period 2π, with scalar product

$$(\varphi, \psi)_{\mathcal{H}} := \int_0^{2\pi} dt \int_{\mathbb{R}^3} dx\ \varphi(t, x)\overline{\psi(t, x)}. \tag{1.8}$$

Hence, let us define the enlarged scattering operator, $\mathcal{S}$,

$$(\mathcal{S}\varphi)(t, x) := S(t)\varphi(t, x),\ \varphi \in \mathcal{H}. \tag{1.9}$$

$\mathcal{S}$ is unitary on $\mathcal{H}$. Let us denote,

$$F_0 := -i\frac{\partial}{\partial t} + H_0. \tag{1.10}$$

Taking the derivative with respect to t in (1.9) and using (1.6) see that formally,

$$F_0\, \mathcal{S} = \mathcal{S}\, F_0. \tag{1.11}$$

Equation (1.1) can be considered as an approximation to the interaction of a quantum particle with an external quantum field. In this approximation $-i\frac{\partial}{\partial t}$ is the energy operator for the external quanta (note that the spectrum of $-i\frac{\partial}{\partial t}$ is $\mathbb{Z}$) and $\mathcal{H}$ is the state space for the quanta and the particle. Furthermore, F_0 is the total free energy operator, and $\mathcal{S}$ is the scattering operator for the quanta and the particle. F_0 is usually called the free quasi–Hamiltonian or the free Floquet Hamiltonian. The commutation relation (1.11) tells us that the free quasi–energy is conserved in the scattering experiment. In other words, in the scattering experiment the particle can gain or loose energy only by absorbing or emitting quanta of the external field. It is quite remarkable that on spite of the fact that we consider the external field as a classical time–dependent potential, V, the emitted and absorbed energy is quantized [29]. These considerations make it natural to define the scattering matrix for $\mathcal{S}$ as the operator that is obtained by diagonalizing $\mathcal{S}$ in a spectral representation of F_0

that expresses the quanta and particle content of F_0 in a natural way. This spectral representation is constructed as follows. For any $n \in \mathbb{Z}$ denote $O_n = (n, \infty)$ and

$$\hat{\mathcal{H}} := \oplus_{n=-\infty}^{\infty} L^2(O_n, L^2\left(S_1^2\right), d\lambda). \tag{1.12}$$

Then, by taking Fourier series in t and Fourier transform in x we obtain an unitary operator, $\mathcal{F}_0$, from $\mathcal{H}$ onto $\hat{\mathcal{H}}$, such that

$$\mathcal{F}_0 F_0 \mathcal{F}_0^{-1} = \lambda, \tag{1.13}$$

is the operator of multiplication by the quasi–energy λ on $\hat{\mathcal{H}}$. Moreover, for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$ we denote,

$$\hat{\mathcal{H}}(\lambda) := \oplus_{m=-\infty}^{n} L^2\left(S_1^2\right), \tag{1.14}$$

where n is the only integer such that, $n < \lambda < n+1$. Note that,

$$\hat{\mathcal{H}} = \oplus \int_{-\infty}^{+\infty} \hat{\mathcal{H}}(\lambda) d\lambda. \tag{1.15}$$

We designate,

$$\hat{\mathcal{S}} := \mathcal{F}_0 \mathcal{S} \mathcal{F}_0^{-1}. \tag{1.16}$$

Hence, we prove that there is an unitary operator, $\hat{\mathcal{S}}(\lambda)$, on $\hat{\mathcal{H}}(\lambda)$ (see (4.78), (4.87) and (4.88)) such that,

$$\left(\hat{\mathcal{S}}\varphi\right)(\lambda) = \hat{\mathcal{S}}(\lambda)\varphi(\lambda), \tag{1.17}$$

for any $\varphi = \varphi(\lambda) \in \oplus \int_{-\infty}^{+\infty} \hat{\mathcal{H}}(\lambda) d\lambda$. Furthermore, $\hat{S}(\lambda) = I + T(\lambda)$, where $T(\lambda)$ is an integral operator in $\hat{\mathcal{H}}(\lambda)$. $\hat{S}(\lambda)$ is the scattering matrix and the Hilbert–Schmidt integral kernels of $T(\lambda)$ are the scattering amplitudes, both at a fixed quasi–energy λ. The fact that $\hat{S}(\lambda)$ is an operator on $\hat{\mathcal{H}}(\lambda)$ exhibits the multi–channel nature of our scattering process, where quanta of the external field are emitted or absorbed by the particle.

The potentials $V(t, \cdot) \in L^{3/2}$ that we consider are so singular that the Hamiltonian $H(t)$ can not be defined as an operator sum and we have to use quadratic form methods. Note, however, that defining the Hamiltonian by quadratic form methods is quite natural from the physical point of view,

as what is measured experimentally are the transition probabilities $(H(t)\phi, \psi)$. Once we realize -in mathematical as well as in physical grounds- that the Hamiltonian has to be defined by quadratic form methods it is natural to assume that the potential factorizes as $V = V_1 V_2$, and to give our conditions on $V_j, j = 1, 2$. This is also convenient since the singularities of the potential make it necessary to use the factorization method to solve the direct scattering problem.

To motivate our conditions it is instructive to first consider the case of time-independent potentials. By Sobolev's imbedding theorem $W_1 \subset L^6$. Moreover, multiplication by $V_j \in L^3, j = 1, 2$, is a bounded operator from L^6 into L^2. As W_1 is the quadratic form domain of the Laplacian, $V := V_1 V_2$ is infinitesimally quadratic form bounded with respect to H_0. This makes it possible to define the Hamiltonian $H_0 + V$ by quadratic form methods. If $V \in L^{3/2}$ we can take, $V_1 := |V|^{1/2}$, and $V_2 := |V|^{1/2}\text{sign}V$. Each of the inclusions above is sharp, and this is the reason why $3/2$ is the critical exponent for the singularities of the potential. In the time-periodic case we give conditions on V that allow us to adapt these estimates in a natural way. We assume that V factorizes as follows,

$$V(t,x) = V_1(t,x)\, V_2(t,x), \tag{1.18}$$

where

$$V_1(t,x) = \sum_{m=-\infty}^{+\infty} e^{imt} V_{1,m}(x),\; V_2(t,x) = \left(\sum_{m=-\infty}^{+\infty} e^{imt} V_{2,m}(x)\right) V_3(t,x), \tag{1.19}$$

with

$$\sum_{m=-\infty}^{+\infty} \|V_{j,m}\|_{L^3} < \infty, j = 1, 2, \tag{1.20}$$

$V_3(t,x) \in L^\infty\left(\mathbb{R}^4\right)$, and $V_3(t+2\pi, x) = V_3(t,x)$. One possible choice is $V_1(t,x) := |V(t,x)|^{1/2}$, $V_2(t,x) := |V(t,x)|^{1/2}\text{sign}V(t,x)$. As mentioned above, in the time independent case these conditions are satisfied if $V \in L^{3/2}$. Note that (1.20) is a condition on the regularity in time of $\sum_{m=-\infty}^{+\infty} e^{imt} V_{j,m}(x), j = 1, 2$. In fact, there is a trade off between the singularities of the potential in space and its regularity in time. We take advantage of this trade off by assuming that the potential is a product of a bounded function, that is only measurable in time, and of two factors that can have singularities in space of type L^3, but that are regular enough in time. To solve the inverse problem we further assume in Theorem 1.1 that the potential decays exponentially.

The perturbed quasi–Hamiltonian, F, is a self–adjoint extension of $F_0 + V$. Our main result is the following theorem.

THEOREM 1.1. *Suppose that V is real valued and that it factorizes as, $V(t,x) = V_1(t,x)\, V_2(t,x)$,*

where for some $\delta_0 > 0, V_1(t,x) = e^{-\delta_0|x|} \sum_{m=-\infty}^{+\infty} e^{imt} V_{1,m}(x), V_2(t,x) = e^{-\delta_0|x|} \left(\sum_{m=-\infty}^{+\infty} e^{imt} V_{2,m}(x) \right) V_3(t,x)$, *with*
$\sum_{m=-\infty}^{+\infty} \|V_{j,m}\|_{L^3} < \infty, j = 1,2,\ V_3(t,x) \in L^\infty(\mathbb{R}^4)$, *and* $V_3(t+2\pi, x) = V_3(t,x), t \in \mathbb{R}$. *Then, the scattering matrix,* $\hat{\mathcal{S}}(\lambda)$, *known at any fixed* $\lambda \in \mathbb{R}\backslash\mathbb{Z}$ *that is not an eigenvalue of* F, *determines uniquely the potential* V.

As we show in Theorem 4.3 if V is small enough, F has no eigenvalues. For a result on the absence of eigenvalues when V is repulsive see [40], [41]. For the exponential decay of quasi–stationary states see [46]. For the uniqueness of the inverse scattering problem for N–body systems with time–dependent potentials when the high–energy limit of the scattering operator is known see [38].

The paper is organized as follows.

In Section 2 we construct the spectral representation of F_0. Furthermore, we state results on the limiting absorption principle (LAP) for F_0 that are an immediate consequence of the corresponding results for the Laplacian in $\mathbb{R}^3$.

In Section 3 we construct the unitary propagator $U(t,t_0)$. Here we follow the method of Yajima [45], [47]. Actually, our result is an extension of Yajima's [45], [47] to the critical singularity $L^{3/2}$. This is possible by the use of the end–point Strichartz estimate [20].

In Section 4 we prove the LAP for F, and we establish the existence and completeness of the wave operators. Here we extend the previous results of Yajima [43] and Howland [10], to the critical singularity $L^{3/2}$. Furthermore, we construct the scattering matrix. We use the Howland–Floquet method [9], [43], [10] and the Kato–Kuroda scattering theory [19], [22].

In Section 5 we prove Theorem 1.1 adapting to our case the proof of uniqueness at a fixed energy for bounded exponentially decreasing time–independent potentials given in [34] by Uhlmann and Vasy. Here the estimates and the generalized limiting absorption principle for Faddeev's Green operator that we obtained in [36] play an essential role. For other results in uniqueness at a fixed energy for exponentially decreasing time–independent potentials see [26], [4] and [12]. For uniqueness at a fixed energy of compactly supported perturbations of a short–range potential see [37]. For perturbed stratified media see [13], [39] and [8].

Finally, we briefly describe the Howland–Floquet method [9], [43], [10] that we use. Let us define the following strongly–continuous unitary groups in $\mathcal{H}$,

$$\left(Y_0(\tau)\varphi\right)(t) := e^{-i\tau H_0}\varphi(t-\tau), \tag{1.21}$$

$$(Y(\tau)\varphi)(t) := U(t, t-\tau)\varphi(t-\tau), \tau \in \mathbb{R}. \tag{1.22}$$

The generators of Y_0 and Y are, respectively, F_0 and F, i.e., $Y_0(\tau) = e^{-i\tau F_0}$, and $Y(\tau) = e^{-i\tau F}$, or in a more precise way, a self–adjoint realization of the free and perturbed quasi–Hamiltonians. Then,

$$\left(e^{i\tau F} e^{-i\tau F_0}\varphi\right)(t) = U(t,0)U(0,t+\tau)e^{-i(\tau+t)H_0}e^{itH_0}\varphi(t). \tag{1.23}$$

Hence, defining

$$\mathcal{W}_{\pm} := \mathrm{s} - \lim_{\tau\to\pm\infty} e^{i\tau F} e^{-i\tau F_0}, \tag{1.24}$$

and using (1.5), we obtain that

$$(\mathcal{W}_{\pm}\varphi)(t) = W_{\pm}(t)\varphi(t), \tag{1.25}$$

and then (see (1.9))

$$\mathcal{S} = \mathcal{W}_+^* \mathcal{W}_-. \tag{1.26}$$

This means that we can study the scattering theory for (1.1) by studying the wave operators $\mathcal{W}_{\pm}$. Since now time is another coordinate, we can apply to this extended scattering problem the stationary theory of Kato and Kuroda [19], [22]. Note that $\mathcal{W}_{\pm}$ and $\mathcal{S}$ are, respectively, the wave and the scattering operators for the quanta and the particle.

For applications to quantum mechanics and to atomic physics of the scattering problem discussed in this paper see [11] and [25].

2 The Free Quasi–Hamiltonian

We define F_0 as the following self–adjoint operator in $\mathcal{H}$,

$$(F_0\varphi)(t,x) := \left(-i\frac{\partial}{\partial t} + H_0\right)\varphi(t,x), \tag{2.1}$$

with domain,

$$D(H_0) := \{\varphi \in \mathcal{H} : (-i\frac{\partial}{\partial t} + H_0)\varphi \in \mathcal{H}\}, \tag{2.2}$$

with the derivatives in distribution sense. By $\mathcal{F}_s$ we denote the Fourier series,

$$(\mathcal{F}_s\varphi)_m := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \varphi(t) e^{-imt} dt, \tag{2.3}$$

as an unitary operator from $L^2(T)$ onto ℓ^2 and by $\mathcal{F}_T$ the Fourier transform,

$$(\mathcal{F}_T\varphi)(k) := \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{+\infty} e^{-ikx}\varphi(x)dx, \tag{2.4}$$

as an unitary operator on L^2. Then,

$$\tilde{\mathcal{F}} := \mathcal{F}_s \times \mathcal{F}_T \tag{2.5}$$

is unitary from $\mathcal{H}$ onto

$$\tilde{\mathcal{H}} := \ell^2(L^2). \tag{2.6}$$

Clearly,

$$F_0 = \tilde{\mathcal{F}}^{-1}(m + k^2)\tilde{\mathcal{F}}, \tag{2.7}$$

and

$$D(F_0) = \{\varphi : (m + k^2)(\tilde{\mathcal{F}}\varphi)_m(k) \in \tilde{\mathcal{H}}\}. \tag{2.8}$$

Then, F_0 is absolutely continuous and its spectrum is $\mathbb{R}$. Define the following unitary operator from $\tilde{\mathcal{H}}$ onto $\hat{\mathcal{H}}$ (see (1.12)),

$$(\mathcal{F}_r\varphi)_m(\lambda, \nu) := \frac{1}{\sqrt{2}}(\lambda - m)^{\frac{1}{4}} \varphi_m\left(\sqrt{\lambda - m}\,\nu\right), \lambda \in (m, \infty), \nu \in S_1^2, \tag{2.9}$$

and designate,

$$\mathcal{F}_0 := \mathcal{F}_r\tilde{\mathcal{F}}. \tag{2.10}$$

Then,

$$\hat{F}_0 := \mathcal{F}_0 F_0 \mathcal{F}_0^{-1} = \lambda, \tag{2.11}$$

is the operator of multiplication by the quasi–energy λ on $\hat{\mathcal{H}}$, i.e., $\mathcal{F}_0$ gives us the spectral representation that we need. Observe that for φ with compact support,

$$(\mathcal{F}_0\varphi)_m(\lambda, \nu) = \int \overline{\phi_m(t, x, \lambda, \nu)}\varphi(t, x) dt\, dx, \tag{2.12}$$

where ϕ_m is the following generalized eigenfunction of F_0,

$$\phi_m(t, x, \lambda, \nu) := \frac{1}{\sqrt{2}} \frac{(\lambda - m)^{\frac{1}{4}}}{(2\pi)^2} e^{imt} e^{i(\lambda-m)^{1/2}\nu\cdot x}. \tag{2.13}$$

For $s \in \mathbb{R}$ let us denote by L^2_s the weighted L^2 space,

$$L^2_s := \left\{\varphi \in \mathcal{D}' : (1 + x^2)^{s/2}\varphi(x) \in L^2\right\}, \tag{2.14}$$

with norm,

$$\|\varphi\|_{L^2_s} = \|(1 + x^2)^{s/2}\varphi(x)\|_{L^2}. \tag{2.15}$$

For $\rho > 0$ let $T(\rho)$ be the bounded trace operator from L^2_s, $s > 1/2$, into $L^2(S^2_1)$ such that,

$$(T(\rho)\varphi)(\nu) = \rho\, (\mathcal{F}_T\varphi)(\rho\, \nu)\ ,\ \varphi \in C^\infty_0. \tag{2.16}$$

$T(\rho)$ has the following properties (see for example [23], pages 4.20 and 4.26).

1)

$$T(0) := \lim_{\rho\downarrow 0} T(\rho) = 0, \tag{2.17}$$

where the limit exists in the operator norm.

2)

$$\|T(\rho)\|_{\mathcal{B}(L^2_s, L^2(S^2_1))} \leq C_s, \rho \geq 0, \tag{2.18}$$

$$\|T(\rho) - T(\rho')\|_{\mathcal{B}(L^2_s, L^2(S^2_1))} \leq C|\rho - \rho'|^{s-1/2}, 1/2 < s < 3/2, \rho, \rho' \geq 0. \tag{2.19}$$

Let us denote

$$\mathcal{H}_s := L^2(T, L^2_s), \;\; s \in \mathbb{R}. \tag{2.20}$$

We define,

$$\left(T_m(\lambda)\varphi\right)(\nu) := \frac{1}{\sqrt{2}} \frac{1}{(\lambda - m)^{1/4}} \left[T((\lambda - m)^{1/2})(\mathcal{F}_s \varphi)_m \right](\nu), \tag{2.21}$$

for $\lambda > m$, and

$$(T_m(\lambda)\varphi)(\nu) = 0, \quad \text{for} \quad \lambda \leq m.$$

Define $\hat{\mathcal{H}}(\lambda)$ as in (1.14) and

$$D(\lambda) := \oplus_{m=-\infty}^{n} T_m(\lambda), \quad n < \lambda < n+1. \tag{2.22}$$

Then, for $\lambda \in \mathbb{R} \backslash \mathbb{Z}, D(\lambda) \in \mathcal{B}(\mathcal{H}_s, \hat{\mathcal{H}}(\lambda)), s > 1/2$, and

$$\|D(\lambda)\|_{\mathcal{B}(\mathcal{H}_s, \hat{\mathcal{H}}(\lambda))} \leq C(1 + |\lambda - n|^{\frac{s-1}{2}}), n < \lambda < n+1. \tag{2.23}$$

Denote,

$$\hat{\mathcal{H}}(\infty) := \oplus_{m=-\infty}^{\infty} L^2\left(S_1^2\right). \tag{2.24}$$

Hence, $\hat{\mathcal{H}}(\lambda) \subset \hat{\mathcal{H}}(\infty)$, with the natural imbedding where we take $\varphi_m \equiv 0$ for $m > \lambda$. $D(\cdot)$ is a locally Hölder continuous function from $\mathbb{R} \backslash \mathbb{Z}$ into $\hat{\mathcal{H}}(\infty)$ with exponent $s - 1/2$, if $1/2 < s < 3/2$. Moreover, if $1 < s < 3/2$ it extends to a Hölder continuous function defined also for λ integer, but the exponent at any integer λ is $\frac{s-1}{2}$.

Let us denote by E_0 the spectral family of F_0. Then,

$$\mathcal{F}_0 E_0(\Delta)\varphi = \chi_\Delta(\lambda) D(\lambda)\varphi, \varphi \in \mathcal{H}_s, s > 1/2, \tag{2.25}$$

for any Borel set Δ, and where χ_O denotes the characteristic function of any set $O \subset \mathbb{R}$.

Let P_m be the following orthogonal projection operator,

$$P_m\varphi = \frac{e^{imt}}{2\pi}\int_0^{2\pi} e^{-imt}\varphi(t,x)\,dt. \tag{2.26}$$

We have that,

$$\mathcal{H} = \oplus_{m=-\infty}^{\infty}\mathcal{H}_m, \text{ where } \mathcal{H}_m := P_m\mathcal{H}. \tag{2.27}$$

For $z \in \mathbb{C}_\pm$ denote,

$$R_0(z) := (F_0 - z)^{-1}, \tag{2.28}$$

and

$$r_0(z) := (H_0 - z)^{-1}. \tag{2.29}$$

Clearly,

$$R_0(z) = \oplus_{m=-\infty}^{\infty} r_0(z-m)P_m. \tag{2.30}$$

Let $W_\alpha, \alpha \geq 0$, be the Sobolev space,

$$W_\alpha := \{\varphi \in L^2 : (1+k^2)^{\alpha/2}(\mathcal{F}_T\varphi)(k) \in L^2\}, \tag{2.31}$$

with norm,

$$\|\varphi\|_{W_\alpha} := \|(1+k^2)^{\alpha/2}(\mathcal{F}_T\varphi)(k)\|_{L^2}. \tag{2.32}$$

For $\alpha < 0, W_\alpha$ is the dual of $W_{-\alpha}$ (with the pairing given by the L^2 scalar product). Moreover, define,

$$W_{\alpha,s} := \left\{\varphi \in L^2_s : (1+x^2)^{s/2}\varphi(x) \in W_\alpha\right\}, \tag{2.33}$$

with norm

$$\|\varphi\|_{W_{\alpha,s}} := \|(1+x^2)^{s/2}\varphi(x)\|_{W_\alpha}. \tag{2.34}$$

Finally, we designate,

$$\mathcal{K}_{\alpha,s} := L^2(T, W_{\alpha,s}). \tag{2.35}$$

The following results on the limiting absorption principle (LAP) for F_0 are an immediate consequence of (2.30) and of the well known results on the LAP for H_0, [1], [6], [14], [15], and [23]. The following limits,

$$R_{0,\pm}(\lambda) := \lim_{\varepsilon \downarrow 0} R_0(\lambda \pm i\varepsilon) \tag{2.36}$$

exist in the uniform operator topology on $\mathcal{B}(\mathcal{H}_s, \mathcal{K}_{1,-s})$, $s > 1/2$, for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$. The convergence is uniform for λ in compact sets of $\mathbb{R}\backslash\mathbb{Z}$ and the functions,

$$R_{0,\pm}(\lambda) := \begin{cases} R_0(\lambda), & \Im\lambda \neq 0, \\ \\ R_{0,\pm}(\lambda) & , \Im\lambda = 0, \end{cases} \tag{2.37}$$

defined for $\lambda \in \mathbb{C}_\pm \cup \mathbb{R}\backslash\mathbb{Z}$ with values in $\mathcal{B}(\mathcal{H}_s, \mathcal{K}_{1,-s})$ are analytic for $\Im\lambda \neq 0$ and locally Hölder continuous for $\Im\lambda = 0$, with exponent, γ, satisfying $\gamma < s - 1/2, 1/2 < s < 3/2$. If $s > 1, R_{0,\pm}(\lambda)$ extend to Hölder continuous functions on $\overline{\mathbb{C}_\pm}$ but the exponent of Hölder continuity at $\lambda \in \mathbb{Z}$ satisfies $\gamma < s-1, 1 < s < 3/2$. Furthermore, $R_{0,\pm}(\lambda)$ are bounded operators on $\mathcal{B}(\mathcal{H}_s, \mathcal{K}_{\alpha,-s}), 0 \leq \alpha \leq 1, s > 1/2$, for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$ and if $s > 1$ also at $\lambda = \mathbb{R}$. Moreover, for any $\delta > 0$ there is a constant C_δ such that

$$\|R_{0,\pm}(\lambda)\|_{\mathcal{B}(\mathcal{H}_s, \mathcal{K}_{\alpha,-s})} \leq \frac{C_\delta}{(1 + \inf_n |\lambda - n|)^{\alpha - 1}}, 0 \leq \alpha \leq 1, \tag{2.38}$$

for all $\lambda \in \mathbb{R}$ with $\inf_n \; |\lambda - n| \geq \delta$, and where if $s > 1$ we can take $\delta = 0$. Moreover, the $R_{0,\pm}(\lambda)$ are compact operators from $\mathcal{H}_s$ into $\mathcal{K}_{\alpha,-s}, 0 \leq \alpha < 1$, with λ and s as above.

It follows from (2.25) and the Stone's theorem that,

$$\frac{d}{d\lambda} E_0(\lambda) = \frac{1}{2\pi i}\Big[R_0(\lambda + i0) - R_0(\lambda - i0)\Big] = D^*(\lambda) D(\lambda), \lambda \in \mathbb{R}\backslash\mathbb{Z}, \tag{2.39}$$

as a bounded operator on $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$, $s > 1/2$, and if $s > 1$ also at $\lambda \in \mathbb{Z}$.

3 The Unitary Propagator

Let us define the following class of potentials.

DEFINITION 3.1. *For any interval $I \subset \mathbb{R}$ we denote by $\mathcal{V}(I)$ the class of potentials $V(t,x), t \in I, x \in \mathbb{R}^3$, such that*

$$V(t,x) = V_1(t,x) + V_2(t,x) \tag{3.1}$$

where,

$$V_1(t,x) \in L^\infty(I, L^{3/2}), \tag{3.2}$$

and

$$V_2(t,x) \in L^1(I, L^\infty). \tag{3.3}$$

■

Note that $\mathcal{V}(I)$ is a Banach space with the norm,

$$\|V\|_{\mathcal{V}(I)} := \inf\left\{\|V_1\|_{L^\infty(I,L^{3/2})} + \|V_2\|_{L^1(I,L^\infty)} : V = V_1 + V_2\right\}. \tag{3.4}$$

The operator $(H_0+1)^{-1/2}$ is an integral operator with kernel $G(x-y)$, where G is the Bessel potential that satisfies [30],

$$|G(x)| \leq C_a |x|^{-2} e^{-a|x|}, \text{ for all } a > 1. \tag{3.5}$$

Then, by the Hölder and the generalized Young inequalities [27]

$$\||V_1(t)|^{1/2}(H_0+1)^{-1/2}\|_{\mathcal{B}(L^2)} \leq C\||V_1(t)|^{1/2}\|_{L^3}. \tag{3.6}$$

Consider $g_n \in C_0^\infty(\mathbb{R}^3)$ such that $g_n \to |V_1(t)|^{1/2}$ in the norm of L^3. By the Rellich local compactness theorem $g_n(H_0+1)^{-1/2}$ is compact in L^2, and furthermore, by (3.6)

$$\|(|V_1(t)|^{1/2} - g_n)(H_0+1)^{-1/2}\|_{\mathcal{B}(L^2)} \leq C\||V_1(t)|^{1/2} - g_n\|_{L^3}, \tag{3.7}$$

and it follows that $|V_1(t)|^{1/2}(H_0+1)^{-1/2}$ is compact. Then, the quadratic form,

$$h_t(\varphi, \psi) = (H_0\varphi, \psi) + (V(t)\varphi, \psi), \tag{3.8}$$

with domain W_1 is closed and bounded below. Let $H(t)$ be the associated self–adjoint operator [27] for a.e. $t \in I$.

Let us consider the integral equation associated to (1.1)

$$\varphi(t) = U_0(t - t_0)\varphi_0 - i \int_{t_0}^{t} U_0(t - \tau)V(\tau)\varphi(\tau)d\tau, \tag{3.9}$$

where,

$$U_0(t - t_0) := e^{-i(t-t_0)H_0}. \tag{3.10}$$

We construct below the solutions to (1.1) by solving (3.9). The key issue for this purpose is the following end–point Strichartz estimates. Let us denote,

$$(G_{t_0}\varphi)(t) := -i \int_{t_0}^{t} U_0(t - \tau)\varphi(\tau)d\tau. \tag{3.11}$$

Let I be any interval in $\mathbb{R}$, and denote,

$$L^{p,q}(I) := L^q(I, L^p), 1 \leq p, q \leq \infty. \tag{3.12}$$

The function $\varphi \in L^{p,q}_{loc}(I)$ if $\varphi \in L^{p,q}(I')$ for I' any compact subinterval of I. Then [20],

$$e^{-itH_0} \in \mathcal{B}(L^2, L^{6,2}(I)), \tag{3.13}$$

and for $t_0 \in I$

$$G_{t_0} \in \mathcal{B}(L^{6/5,2}(I), L^{6,2}(I)) \cap \mathcal{B}(L^{6/5,2}(I), C_b(I, L^2)) \cap \mathcal{B}(L^{2,1}(I), L^{6,2}(I)). \tag{3.14}$$

Moreover, trivially,

$$e^{-itH_0} \in \mathcal{B}(L^2, C_b(I, L^2)), \tag{3.15}$$

$$G_{t_0} \in \mathcal{B}(L^{2,1}(I), C_b(I, L^2)), \tag{3.16}$$

where $C_b(I, L^2)$ denotes the continuous and bounded functions from I into L^2. Furthermore, the bounds on (3.13) - (3.16) can be taken uniform on t_0 and I. Let us designate

$$\mathcal{A}(I) := C_b(I, L^2) \cap L^{6,2}(I), \tag{3.17}$$

with norm

$$\|\varphi\|_{\mathcal{A}(I)} := \max\left[\|\varphi\|_{C_b(I,L^2)}, \|\varphi\|_{L^{6,2}(I)}\right], \tag{3.18}$$

and

$$\mathcal{A}'(I) := L^{2,1}(I) + L^{6/5,2}(I), \tag{3.19}$$

with norm,

$$\|\varphi\|_{\mathcal{A}'(I)} := \inf\left\{\|\varphi_1\|_{L^{2,1}(I)} + \|\varphi_2\|_{L^{6/5,2}(I)} : \varphi = \varphi_1 + \varphi_2\right\}. \tag{3.20}$$

We prepare the following result.

LEMMA 3.2. *Given $\varepsilon > 0$ there is a $\delta > 0$ such that for any interval $I' \subset I$, with $|I'| \leq \delta$,*

$$\|V\|_{\mathcal{V}(I')} \leq \varepsilon. \tag{3.21}$$

Proof: Take $V_{1,m} \in L^\infty(I, L^{3/2} \cap L^\infty)$ such that

$$\|V_1 - V_{1,m}\|_{L^\infty(I,L^{3/2})} \leq \frac{\varepsilon}{2}, \tag{3.22}$$

and denote

$$V_m = V_{1,m} + V_2. \tag{3.23}$$

Then,

$$\|V - V_m\|_{\mathcal{V}(I)} \leq \frac{\varepsilon}{2}. \tag{3.24}$$

Moreover,

$$\|V_m\|_{\mathcal{V}(I')} \leq \|V_m\|_{L^1(I',L^\infty)} \leq \frac{\varepsilon}{2}, \tag{3.25}$$

if $|I'| \leq \delta$ for δ small enough.

■

By Hölder's inequality,

$$\|V\|_{\mathcal{B}(\mathcal{A}(I),\mathcal{A}'(I))} \leq \|V\|_{\mathcal{V}(I)}. \tag{3.26}$$

Moreover, by (3.14) and (3.16),

$$G_{t_0} \in \mathcal{B}(\mathcal{A}'(I), \mathcal{A}(I)) \tag{3.27}$$

with bound uniform on t_0 and on I. Denote,

$$Q_{t_0}\varphi := G_{t_0} V\varphi. \tag{3.28}$$

Hence, by (3.26) and (3.27)

$$\|Q_{t_0}\|_{\mathcal{B}(\mathcal{A}(I))} \leq C\|V\|_{\mathcal{V}(I)}, \tag{3.29}$$

where the constant C is independent of t_0. In consequence, by Lemma 3.2 for any $t_0 \in I$ there is a $\delta > 0$ such that for $I' = [t_0 - \delta/2, t_0 + \delta/2]$, Q_{t_0} is a contraction on $\mathcal{A}(I')$ and then (3.9) has an unique solution on $\mathcal{A}(I')$ given by

$$\varphi(t) = (I - Q_{t_0})^{-1}\, T_{t_0}\varphi_0,\ t \in I', \tag{3.30}$$

where

$$T_{t_0}\varphi_0 = U_0(t - t_0)\varphi_0. \tag{3.31}$$

The following Theorem is now proven as in the proof of Theorem 1 of [47] (see also [45]) by extending the solution given by (3.30) to $t \in I$ in successive steps of length δ. As by Sobolev's theorem $L^{6/5}$ is continuously imbedded in W_{-2}, we have that $V \in \mathcal{B}(L^6, W_{-2})$, and then $H_0 + V \in \mathcal{B}(L^2 \cap L^6, W_{-2})$. Moreover, for $\varphi \in D(H(t)) \cap L^6$,

$$H(t)\varphi = H_0\varphi + V(t)\varphi. \tag{3.32}$$

We also use the notation $H(t)$ for $H_0 + V(t)$ when viewed as a bounded operator from $L^2 \cap L^6$ into W_{-2}.

THEOREM 3.3. *Suppose that* V *satisfies* $V(t + 2\pi, x) = V(t, x)$, $t \in \mathbb{R}, x \in \mathbb{R}^3$ *and that* $V \in \mathcal{V}([0, 2\pi])$*. Then, there exists a unique propagator* $U(t, t_0), (t, t_0) \in \mathbb{R}^2$ *with the following properties.*

1) $U(t, t_0)$ *is unitary in* L^2 *with* $U(t, t_1)U(t_1, t_0) = U(t, t_0), t_0, t_1, t \in \mathbb{R}$.

2) $U(\cdot, \cdot)$ *is a strongly–continuous function from* $\mathbb{R}^2$ *into* $\mathcal{B}(L^2)$.

3) $U(t + 2\pi, t_0 + 2\pi) = U(t, t_0), t, t_0 \in \mathbb{R}$.

4) For any $t_0 \in \mathbb{R}, \varphi \in L^2, U(\cdot, t_0)\varphi \in L^{6,2}_{loc}(\mathbb{R})$ *and it satisfies the equation*

$$U(t, t_0)\varphi_0 = U_0(t - t_0)\varphi_0 - i \int_{t_0}^{t} U_0(t - \tau)V(\tau)U(\tau, t_0)\varphi_0 \, d\tau. \tag{3.33}$$

5) There is a constant C *such that for all* $t_0 \in \mathbb{R}, \varphi_0 \in L^2$ *and all bounded intervals* $I \subset \mathbb{R}$,

$$\|U(\cdot, t_0)\varphi_0\|_{L^{6,2}(I)} \leq C(1 + |I|)^{1/2} \|\varphi_0\|_{L^2}. \tag{3.34}$$

6) For any $t_0 \in \mathbb{R}$ *and* $\varphi_0 \in L^2, U(\cdot, t_0)\varphi_0$ *is a* W_{-2}*– valued, absolutely-continuous function and it satisfies the equation (1.1),*

$$i\frac{\partial}{\partial t} U(t, t_0)\varphi_0 = H(t)U(t, t_0)\varphi_0. \tag{3.35}$$

■

Theorem 3.3 extends the results of [45], [47] to the critical singularity $L^{3/2}$. We could consider the problem of the regularity of the propagator as in [45] and [47]. Also, as in Theorem 1 of [47] we could study the case of $V \in L^\infty(\mathbb{R}, L^{3/2}) + L^1(\mathbb{R}, L^\infty)$. We do not go in these directions here. For other results on the unitary propagator for time–dependent potentials see [16], [18], [27], [32], and the references quoted in these works.

4 The Limiting Absorption Principle

In this section we always assume that V is real valued, and that $V(t,x) = V_1(t,x)\, V_2(t,x)$, where $V_1(t,x) = (1+|x|)^{-(1+\epsilon)/2} \sum_{m=-\infty}^{+\infty} e^{imt} V_{1,m}(x)$, $V_2(t,x) = (1+|x|)^{-(1+\epsilon)/2} \left(\sum_{m=-\infty}^{+\infty} e^{imt} V_{2,m}(x) \right) V_3(t,x)$, with,

$$\sum_{m=-\infty}^{+\infty} \|V_{j,m}\|_{L^3} < \infty, j = 1,2,\ V_3(t,x) \in L^\infty(\mathbb{R}^4), \text{and} V_3(t+2\pi, x) = V_3(t,x), t \in \mathbb{R}, \tag{4.1}$$

for some $\epsilon > 0$. We could also add to V a bounded short–range term, but for simplicity, and since our aim is to solve the inverse problem for exponentially decreasing potentials, we will not do so. Since $H(t)$ is defined as a quadratic form, i.e., the perturbation $V(t)$ is only form bounded with respect to H_0, we find it convenient to use the Kato–Kuroda theory [19], [22] with the factorization method.

Let us denote by $\chi_1(t,x)$ the characteristic function of the support of $V_1(t,x)$. We define,

$$q_1(t,x) := V_1(t,x) + e^{-x^2}(1 - \chi_1(t,x)), \tag{4.2}$$

$$q_2(t,x) := V_2(t,x) \chi_1(t,x). \tag{4.3}$$

Let A and B be the following maximal operators of multiplication in $\mathcal{H}$,

$$A\varphi := q_1(t,x)\, \varphi(t,x), \tag{4.4}$$

$$B\varphi := q_2(t,x)\, \varphi(t,x). \tag{4.5}$$

Estimating as in (3.6) we prove that A and B are bounded from $\mathcal{K}_{1,0}$ into $\mathcal{H}$. Observe that,

$$V = BA = AB \in \mathcal{B}\left(\mathcal{K}_{1,0}, \mathcal{K}_{-1,0}\right). \tag{4.6}$$

We define $q_j(t,x), j = 1,2$, as above only to simplify some of the proofs below. With this definition the range of A is dense in $\mathcal{H}$. Let us denote by $R(z) := (F - z)^{-1}, \Im z \neq 0$, the resolvent of F. By functional calculus for $\Im z > 0$,

$$R(z) = i \int_0^\infty e^{iz\tau} Y(\tau) d\tau. \tag{4.7}$$

Then,

$$(R(z)\varphi)(t) = i\int_{-\infty}^{t} e^{izt}U(t,\tau)e^{-iz\tau}\varphi(\tau)d\tau. \tag{4.8}$$

By (4.8)

$$\|(q_j R(z)\varphi)(t)\|_{L^2} \le \int_{-\infty}^{2\pi} \|q_j U(t,\tau)e^{-iz\tau}\varphi(\tau)\|_{L^2}\, d\tau, 0 \le t \le 2\pi, \tag{4.9}$$

and by (3.34) and Hölder's inequality,

$$\|q_j R(z)\varphi\|_{\mathcal{H}} \le C\|q_j\|_{L^{3,\infty}} \int_{-\infty}^{2\pi} e^{\Im z\tau}\|\varphi(\tau)\|_{L^2}\, d\tau \le \frac{C}{\Im z}\, \|q_j\|_{L^{3,\infty}}\, \|\varphi\|_{\mathcal{H}}. \tag{4.10}$$

Then, for $\Im z > 0, A\,R_0(z), BR_0(z), AR(z)$, and $BR(z)$ are bounded in $\mathcal{H}$. We prove that they are also bounded in $\mathcal{H}$ for $\Im z < 0$ in a similar way. It follows from (3.33), (4.8) and a simple calculation (see the proof of Lemma 3.3 of [43]) that,

$$R(z) = R_0(z) - (BR_0(\bar{z}))^* AR(z), \Im z \ne 0. \tag{4.11}$$

This implies, in particular, that F is an extension of $F_0 + BA = F_0 + AB$.

As is well known, $r_0(z)$ is an integral operator with kernel $h_0(\sqrt{z}|x-y|)$, where

$$h_0(\sqrt{z}|x|) := \frac{1}{4\pi}\frac{e^{i\sqrt{z}|x|}}{|x|}, \tag{4.12}$$

where we take the branch of the square root with $\Im\sqrt{z} \ge 0,\ z \in \mathbb{C}$.

LEMMA 4.1. *Let f and g satisfy,*

$$f(t,x) = \sum_{m=-\infty}^{\infty} e^{imt} f_m(x), \sum_{m=-\infty}^{+\infty} \|f_m\|_{L^3} < \infty, \tag{4.13}$$

$$g(t,x) = \sum_{m=-\infty}^{\infty} e^{imt} g_m(x), \sum_{m=-\infty}^{\infty} \|g_m\|_{L^3} < \infty. \tag{4.14}$$

Then,

$$J_{\pm}(\lambda) := fR_{0,\pm}(\lambda)g, \lambda \in \overline{\mathbb{C}^{\pm}} \tag{4.15}$$

are compact operators on $\mathcal{H}$. The $\mathcal{B}(\mathcal{H})$-valued functions $J_\pm$ are analytic for $\lambda \in \mathbb{C}^\pm$ and continuous for $\lambda \in \overline{\mathbb{C}^\pm}$. Furthermore,

$$\lim_{|\Im\lambda| \to \infty} J_\pm(\lambda) = 0, \tag{4.16}$$

where the limit holds in the operator norm in $\mathcal{B}(\mathcal{H})$.

Proof: Denote,

$$\hat{J}_\pm(\lambda) := \mathcal{F}_s f\, R_{0,\pm}(\lambda)\, g\, \mathcal{F}_s^{-1}. \tag{4.17}$$

It is enough to prove that $\hat{J}_\pm(\lambda)$ have the properties stated on the Lemma as an operator on $\tilde{\mathcal{H}} := \ell^2(L^2)$. By (2.30) and (4.12),

$$(\hat{J}_\pm(\lambda)\varphi)_n = \sum_{m=-\infty}^{\infty} d_{\pm,n,m}(\lambda)\, \varphi_m, \tag{4.18}$$

where $d_{\pm,n,m}(\lambda)$ is the integral operator on L^2 with kernel

$$d_{\pm,n,m}(\lambda, x, y) := \sum_{\ell=-\infty}^{\infty} f_{n-\ell}(x)\, h_0(\sqrt{\lambda - \ell}\, |x-y|) g(y)_{\ell-m}. \tag{4.19}$$

By Hölder's and generalized Young's inequalities [27], $d_{\pm,n,m}(\lambda)$ are bounded in L^2 for $\lambda \in \overline{\mathbb{C}}_\pm$ and,

$$\|d_{\pm,n,m}(\lambda)\|_{\mathcal{B}(L^2)} \leq C \sum_{\ell=-\infty}^{\infty} \|f_{n-\ell}\|_{L^3} \|g_{\ell-m}\|_{L^3}, \tag{4.20}$$

and since

$$\sup_n \sum_{m=-\infty}^{\infty} \|d_{\pm,n,m}(\lambda)\|_{\mathcal{B}(L^2)} \leq C \left(\sum_{\ell=-\infty}^{\infty} \|f_\ell\|_{L^3} \right) \sum_{r=-\infty}^{\infty} \|g_r\|_{L^3}, \tag{4.21}$$

$$\sup_m \sum_{n=-\infty}^{\infty} \|d_{\pm,n,m}(\lambda)\|_{\mathcal{B}(L^2)} \leq C \left(\sum_{\ell=-\infty}^{\infty} \|f_\ell\|_{L^3} \right) \sum_{r=-\infty}^{\infty} \|g_r\|_{L^3}, \tag{4.22}$$

we have that,

$$\|\hat{J}_\pm(\lambda)\|_{\mathcal{B}(\tilde{\mathcal{H}})} \leq C \left(\sum_{\ell=-\infty}^{\infty} \|f_\ell\|_{L^3} \right) \sum_{r=-\infty}^{\infty} \|g_r\|_{L^3}. \tag{4.23}$$

Let $h \in C_0^\infty(\mathbb{R}^3)$ satisfy $\int h(x)dx = 1$, and denote $h_\ell(x) = \ell^3 h(\ell x)$. We designate, $f_{m,r}(x) := f_m(x)$ if $|x| \leq r$, $f_{m,r}(x) = 0$ if $|x| \geq r$, and

$$f_m^{(\ell,r)}(x) = \int h_\ell(x-y) f_{m,r}(y)\, dy \in C_0^\infty(\mathbb{R}^3), \tag{4.24}$$

$$g_m^{(\ell,r)}(x) = \int h_\ell(x-y) g_{m,r}(y)\, dy \in C_0^\infty(\mathbb{R}^3), \tag{4.25}$$

with $g_{m,r}$ defined as $f_{m,r}$. $f_m^{(\ell,r)} \to f_m, g_m^{(\ell,r)} \to g_m$, strongly in L^3, as $\ell, r \to \infty$. We define,

$$f^{(\ell,r,p)} := \sum_{|m|\leq p} e^{imt} f_m^{(\ell,r)};\ g^{(\ell,r,p)} := \sum_{|m|\leq p} e^{imt} g_m^{(\ell,r)}. \tag{4.26}$$

As $f^{(l,r,p)}$ and $g^{(l,r,p)}$ are bounded and have compact support in x and as $R_{0,\pm}(\lambda)$ are compact operators from $\mathcal{H}_s$ into $\mathcal{H}_{-s}$, $s > 1/2$, the operators $f^{(l,r,p)}\, R_{0,\pm}(\lambda)\, g^{(l,r,p)}$ are compact in $\mathcal{H}$, for $\lambda \in \overline{\mathbb{C}_\pm}$, and then, we have that,

$$\hat{J}_\pm^{(\ell,r,p)}(\lambda) := \mathcal{F}_s f^{(\ell,r,p)} R_{0,\pm}(\lambda) g^{(\ell,r,p)} \mathcal{F}_s^{-1} \tag{4.27}$$

are compact in $\tilde{\mathcal{H}}$ for $\lambda \in \overline{\mathbb{C}_\pm}$. By (4.23),

$$\lim_{\ell,r,p\to\infty} \hat{J}_\pm^{(\ell,r,p)}(\lambda) = \hat{J}_\pm(\lambda), \tag{4.28}$$

in the uniform operator topology on $\mathcal{B}(\tilde{\mathcal{H}})$, and hence, $\hat{J}_\pm(\lambda)$ are compact for $\lambda \in \overline{\mathbb{C}_\pm}$. Moreover, as F_0 is self–adjoint,

$$\|\hat{J}_\pm^{(\ell,r,p)}(\lambda)\|_{\mathcal{B}(\tilde{\mathcal{H}})} \leq \frac{C}{|\Im\lambda|} \|f^{(\ell,r,p)}\|_{L^\infty\left(T\times\mathbb{R}^3\right)} \|g^{(\ell,r,p)}\|_{L^\infty\left(T\times\mathbb{R}^3\right)}, \tag{4.29}$$

and as the limit in (4.28) is uniform for $\lambda \in \overline{\mathbb{C}_\pm}$, (4.16) holds.

■

Let us denote,

$$Q_{0,\pm}(\lambda) := B R_{0,\pm}(\lambda) A,\ \ \lambda \in \overline{\mathbb{C}_\pm}. \tag{4.30}$$

Since the multiplication operators by χ_1 and by V_3 are bounded on $\mathcal{H}, Q_{0,\pm}(\lambda)$ have all the properties stated in Lemma 4.1. Denote,

$$G_{0,\pm}(\lambda) := I + Q_{0,\pm}(\lambda), \lambda \in \overline{\mathbb{C}_{\pm}}. \tag{4.31}$$

It follows from (4.11) and a simple calculation (see [17], [22] and [43]) that

$$BR(z) = G_{0,\pm}(z)^{-1} BR_0(z), \ z \in \mathbb{C}_{\pm}, \tag{4.32}$$

$$R(z) = R_0(z) - R_0(z) A G_{0,\pm}(z)^{-1} BR_0(z), z \in \mathbb{C}_{\pm}, \tag{4.33}$$

$$R(z)A = R_0(z) A G_{0,\pm}(z)^{-1} \ , \ z \in \mathbb{C}_{\pm}. \tag{4.34}$$

In (4.33), (4.34), by $R_0(z)A$ and $R(z)A$ we actually mean the closure of these operators that are bounded in $\mathcal{H}$. These formulae are first proven for $\Im z$ large enough using (4.16) and then, they are extended to $z \in \mathbb{C}_{\pm}$ by the analyticity of the resolvents. In particular, it follows that $G_{0,\pm}(z)$ is invertible in $\mathcal{B}(\mathcal{H})$ for $z \in \mathbb{C}_{\pm}$. Denote,

$$Q_{\pm}(z) := BR(z)A, \ z \in \mathbb{C}_{\pm}, \tag{4.35}$$

and

$$G_{\pm}(z) := (I - Q_{\pm}(z)), z \in \mathbb{C}_{\pm}. \tag{4.36}$$

Then,

$$G_{\pm}(z) = G_{0,\pm}(z)^{-1}, \ z \in \mathbb{C}_{\pm}. \tag{4.37}$$

To obtain the LAP for F we need to prove that $G_{0,\pm}(\lambda)$ are invertible for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$ if λ is not an eigenvalue of F. This is done extending the well known argument of [1] where the case of time–independent potentials was considered. This was accomplished in [6] and [10], but as they considered the case where the perturbation is relatively bounded, and the factorization method is not needed, we give some details in the Lemma below. We denote by $\sigma_p(F)$ the set of all eigenvalues of F.

LEMMA 4.2. *Suppose that (4.1) holds. Then, $Q_{0,\pm}(\lambda)$ are invertible in $\mathcal{B}(\mathcal{H})$ for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$ if and only if λ is not an eigenvalue of F.*

Proof: We first assume that $\lambda \in (\mathbb{R}\backslash\mathbb{Z}) \cap \sigma_p(F)$, and that $F\varphi = \lambda\varphi$. Taking the adjoint of (4.34) we have that,

$$(I + Q^*_{0,\mp}(\bar{z}))AR(z) = AR_0(z), z \in \mathbb{C}_\pm. \tag{4.38}$$

Then (recall that $D(A) \supset D(F)$),

$$(I + Q^*_{0,\mp}(\bar{z}))A\varphi = AR_0(z)(F - z)\varphi. \tag{4.39}$$

Take now $z = \lambda + i\varepsilon$. Hence,

$$\lim_{\varepsilon\downarrow 0} AR_0(\lambda + i\varepsilon)(F - \lambda - i\varepsilon)\varphi = -\lim_{\varepsilon\downarrow 0} AR_0(\lambda + i\varepsilon)i\varepsilon\varphi = 0, \tag{4.40}$$

where the limit holds in the strong topology of $\mathcal{K}_{-1,(1+\varepsilon)/2}$. Here I use that $A \in \mathcal{B}(\mathcal{H}, \mathcal{K}_{-1,(1+\varepsilon)/2})$ and that $\varepsilon R_0(\lambda + i\varepsilon) \to 0$ as $\varepsilon \to 0$, in the strong topology in $\mathcal{H}$ because F_0 has no eigenvalues. Then,

$$(I + Q^*_{0,\pm}(\lambda))A\varphi = 0, \tag{4.41}$$

and as $A\varphi \in \mathcal{H}$, $(I + Q^*_{0,\pm}(\lambda))$ are not invertible. Since $Q_{0,\pm}(\lambda)$ are compact it follows that $(I + Q_{0,\pm}(\lambda))$ are not invertible in $\mathcal{B}(\mathcal{H})$.

Suppose now that $(I + Q_{0,\pm}(\lambda))$ are not invertible in $\mathcal{B}(\mathcal{H})$. Then, also $(I + Q^*_{0,\pm}(\lambda))$ are not invertible and there are $w_\pm \in \mathcal{H}$ such that

$$w_\pm + AR_{0,\pm}(\lambda)Bw_\pm = 0, w_\pm \neq 0. \tag{4.42}$$

Taking the inner product of (4.42) with $BR_{0,\pm}(\lambda)Bw_\pm$ we obtain that

$$(w_\pm, BR_{0,\pm}(\lambda)Bw_\pm) + (AR_{0,\pm}(\lambda)Bw_\pm, BR_{0,\pm}(\lambda)Bw_\pm) = 0. \tag{4.43}$$

Taking the imaginary part of (4.43) and using (2.39) we have that

$$D(\lambda)Bw_\pm = 0. \tag{4.44}$$

Note that by (2.39) and Lemma 4.1, $D(\lambda)B \in \mathcal{B}(\mathcal{H}, \hat{\mathcal{H}}(\lambda))$. Denote,

$$\varphi_\pm := -R_{0,\pm}(\lambda)Bw_\pm. \tag{4.45}$$

$\varphi_\pm \neq 0$, because otherwise $w_\pm = A\varphi_\pm = 0$. Designate, $\varphi_{\pm,m} := (\mathcal{F}_s \varphi_\pm)_m$, $(Bw_\pm)_m := (\mathcal{F}_s B w_\pm)_m$. Then, by (2.30),

$$\varphi_{\pm,m} = -r_{0,\pm}(\lambda - m)(Bw_\pm)_m. \tag{4.46}$$

The following statements are a slight extension of the results of [1] and [6] that consider the case of $\varphi \in W_{s,0}$. They are proven as in [1], [6].

1) Let $c > 0$ and $s \in \mathbb{R}$. Then, for some constant C, for all $\varphi \in W_{s,-1}$,

$$\left\| \frac{\varphi(k)}{k^2 + \lambda^2} \right\|_{W_{s,0}} \leq \frac{C}{\lambda} \left\| \varphi \right\|_{W_{s,-1}}, \lambda > c. \tag{4.47}$$

2) Let $c > 0$ and $s > 1/2$. Then, for some constant C, for all $\varphi \in W_{s,-1}$ with $\varphi(k)|_{|k|=\lambda} = 0$ in trace sense,

$$\left\| \frac{\varphi(k)}{k^2 - \lambda^2} \right\|_{W_{s-1,0}} \leq C \left\| \varphi \right\|_{W_{s,-1}}, \lambda > c. \tag{4.48}$$

For $\lambda - m < 0$ we obtain from (4.46) and (4.47), with $s = \frac{1+\varepsilon}{2}$,

$$\|\varphi_{\pm,m}\|_{L^2_{2s}} \leq \frac{C}{|\lambda - m|^{1/2}} \|(Bw_\pm)_m\|_{W_{-1,2s}}, \tag{4.49}$$

and when $\lambda - m > 0$ by (4.46) and (4.48),

$$\|\varphi_{\pm,m}\|_{L^2_{2s-1}} \leq C \|(Bw_\pm)_m\|_{W_{-1,2s}}. \tag{4.50}$$

By Lemma 4.1 and (4.42) $w_\pm \in \mathcal{H}_s$, and,

$$\|w_\pm\|_{\mathcal{H}_s} \leq C \|w_\pm\|_{\mathcal{H}_{-s}}. \tag{4.51}$$

Equations (4.49), (4.50) and (4.51) imply that

$$\|\varphi_\pm\|_{\mathcal{H}_{2s-1}} \leq C \|Bw_\pm\|_{\mathcal{K}_{-1,2s}} \leq C \|w_\pm\|_{\mathcal{H}_s} \leq C \|w_\pm\|_{\mathcal{H}_{-s}}, \; 2s - 1 = \varepsilon > 0. \tag{4.52}$$

We now prove that the $\varphi_\pm$ are eigenvectors of F with eigenvalue λ. Taking the adjoint of (4.34) we have that

$$AR(z) = (I + AR_0(z)B)^{-1} AR_0(z), z \in \mathbb{C}_\pm. \tag{4.53}$$

Taking the inverse of (4.53) and multiplying the result by A we obtain

$$(F - z) = (F_0 - z)A^{-1}(I + AR_0(z)B)A, z \in \mathbb{C}_\pm. \tag{4.54}$$

Taking the limit $z \to \lambda$,

$$F - \lambda = (F_0 - \lambda)A^{-1}(I + AR_{0,\pm}(\lambda)B)A. \tag{4.55}$$

Then, since $\varphi_\pm \in D(A)$ and $A\varphi_\pm = w_\pm$, it follows from (4.42) and (4.55) that $\varphi_\pm \in D(F)$ and $F\varphi_\pm = \lambda\varphi_\pm$.

■

The LAP for F follows now from the LAP for F_0 (see Section 2) (4.33) and Lemma 4.2. We state the results in the following theorem.

THEOREM 4.3. *Suppose that (4.1) holds. Then, the following limits*

$$R_\pm(\lambda) := \lim_{\varepsilon \downarrow 0} R(\lambda \pm i\varepsilon), \tag{4.56}$$

exist in the uniform operator topology on $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s}), s = \frac{1+\varepsilon}{2}$, for $\lambda \in \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F)$, where $\sigma_p(F)$ denotes the set of eigenvalues of F. Furthermore,

$$R_\pm(\lambda) = R_{0,\pm}(\lambda) - R_{0,\pm}(\lambda)A\,G_{0,\pm}(\lambda)^{-1}BR_{0,\pm}(\lambda). \tag{4.57}$$

The functions

$$R_\pm(\lambda) := \begin{cases} R(\lambda), & \lambda \in \mathbb{C}_\pm, \\ R_\pm(\lambda), & \lambda \in \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F), \end{cases} \tag{4.58}$$

with values in $\mathcal{B}(\mathcal{H}_s, \mathcal{H}_{-s})$ are analytic for $\lambda \in \mathbb{C}_\pm$ and continuous for $\lambda \in \mathbb{C}_\pm \cup \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F)$. Furthermore, F has no singular-continuous spectrum and $\sigma_p(F)\backslash\mathbb{Z}$ consists of finite dimensional eigenvalues that can only accumulate at $\mathbb{Z}$. Moreover, if $\lambda \in \sigma_p(F), \lambda + m \in \sigma_p(F)$ for all $m \in \mathbb{Z}$. Finally, if either $\sum_{m=-\infty}^{\infty} \|V_{j,m}\|_{L^3}$, for $j = 1$, or for $j = 2$ or $\|V_3\|_{L^\infty\left(\mathbb{R}^4\right)}$ is small enough, then, $\sigma_p(F)$ is empty.

Proof: The existence of the limits in (4.56) and the properties of $R_{\pm}$ have already been proven. The fact that F has no singular–continuous spectrum is a consequence of the LAP for F [28]. In the proof of Lemma 4.2 we established a one–to–one correspondence between the kernel of $F-\lambda$ and the kernel of $(I+Q_{0,\pm}(\lambda))$ for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$. But as $Q_{0,\pm}(\lambda)$ are compact the kernel of the later are finite dimensional, and then the non–integer eigenvalues of F have finite multiplicity. Suppose that λ_j are infinite distinct points of $\sigma_p(F)\backslash\mathbb{Z}$, and that $\lim_{j\to\infty} \lambda_j = \lambda_\infty, \lambda_\infty \notin \mathbb{Z}$. By Lemma 4.2 there are w_j with $\|w_j\|_{\mathcal{H}} = 1$ such that

$$w_j = -AR_{0,+}(\lambda_j)Bw_j, j = 1,2,\cdots. \tag{4.59}$$

As $AR_{0,+}(\lambda_j)B$ are compact we can assume (eventually passing to a subsequence) that $w_j \to w_\infty$ strongly in $\mathcal{H}$ with $\|w_\infty\|_{\mathcal{H}} = 1$, and

$$w_\infty = -AR_{0,+}(\lambda_\infty)Bw_\infty. \tag{4.60}$$

Denote by

$$\varphi_j := -R_{0,+}(\lambda_j)Bw_j,\ j = 0,1,2,\ldots,\infty, \tag{4.61}$$

the corresponding sequence of eigenvectors. Then $\varphi_j \neq 0$ and by (4.52)

$$\|\varphi_j\|_{\mathcal{H}} \leq C, j = 1,2,\ldots,\infty. \tag{4.62}$$

Since $w_j \to w_\infty$ strongly in $\mathcal{H}$, by (4.61) $\varphi_j \to \varphi_\infty$ strongly in $\mathcal{H}_{-s}, s = \frac{1+\varepsilon}{2}$, to φ_∞, and then, by (4.62) $\varphi_j \to \varphi_\infty$ weakly in $\mathcal{H}$. But as φ_∞ and φ_j are eigenvectors corresponding to different eigenvalues of F they are orthogonal, and it follows that $\varphi_\infty = 0$, which is a contradiction. The last statement of the theorem is immediate since $e^{-imt}Fe^{imt} = F + m$. Finally, by the proof of Lemma 4.1 if either $\sum_{m=-\infty}^{+\infty} \|V_{j,m}\|_{L^3}$ for $j=1$, or for $j=2$ or $\|V_3\|_{L^\infty\left(\mathbb{R}^4\right)}$ is small enough $\|Q_{0,\pm}(\lambda)\| < 1$, and then, $I + Q_{0,\pm}(\lambda)$ is invertible for all $\lambda \in \mathbb{R}$. Hence (see (4.33) and (4.58)), the LAP for F holds for all λ in $\mathbb{R}$ and this implies that F has pure absolutely-continuous spectrum [28]. In particular, F has no eigenvalues.

■

Theorem 4.3 extends the results of [10] and [43] to the critical singularity $L^{3/2}$. There are a number of results in the LAP for time–dependent Hamiltonians. See for example, [24], [48], where long–range potentials are considered. In [6] the LAP at $\lambda \in \mathbb{Z}$ is also studied. References [24], [48] and [6] consider potentials that are more regular than ours.

It follows from (2.39) and Lemma 4.1 that $D(\lambda)A \in B(\mathcal{H}, \hat{\mathcal{H}}(\lambda))$. Under the assumptions of Theorem 4.3 $G_{\pm}(\lambda)$ extend to continuous functions from $\mathbb{C}_{\pm} \cup \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F)$ into $\mathcal{B}(\mathcal{H})$ and

$$G_{\pm}(\lambda) = G_{0,\pm}(\lambda)^{-1}, \lambda \in \mathbb{C}_{\pm} \cup \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F). \tag{4.63}$$

Denote,

$$D_{\pm}(\lambda) := D(\lambda)AG_{\pm}(\lambda), \lambda \in \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F). \tag{4.64}$$

Let $E(\lambda)$ be the spectral family of F, and let $\mathcal{H}_{ac}(F)$ be the subspace of absolute continuity of F. Then, for any $\varphi \in \mathcal{H}_{ac}(F)$ of the form,

$$\varphi = \sum_{j=1}^{N} E(I_j)A\varphi_j, \bar{I}_j \subset \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F), \tag{4.65}$$

$\bar{I}_j$ compact, $I_j \cap I_k = \phi, j \neq k$, define the operators

$$\mathcal{F}_{\pm}\varphi := \sum_{j=1}^{N} \chi_{I_j}(\lambda)D_{\pm}(\lambda)A\varphi_j. \tag{4.66}$$

Hence, under (4.1) the following facts are proven as in the proof of Theorem 3.11 of [22] (see also the proof of Lemma 7.1 of [35]). The $\mathcal{F}_{\pm}$ defined by (4.66) extend to unitary operators from $\mathcal{H}_{ac}(F)$ onto $L^2(\mathbb{R}, \hat{\mathcal{H}})$ and for any $\varphi \in D(A)$ and any Borel set $I \subset \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F)$,

$$\mathcal{F}_{\pm}E(I)A\varphi = \chi_I(\lambda)D_{\pm}(\lambda)\varphi, \tag{4.67}$$

and if $P_{ac}(F)$ denotes the orthogonal projector onto $\mathcal{H}_{ac}(F)$,

$$FP_{ac}(F) = \mathcal{F}_{\pm}^{-1}\lambda\,\mathcal{F}_{\pm}. \tag{4.68}$$

We now define the wave operators as

$$\mathcal{W}_{\pm} := \mathcal{F}_{\pm}^{*}\mathcal{F}_0. \tag{4.69}$$

The $\mathcal{W}_{\pm}$ are unitary from $\mathcal{H}$ onto $\mathcal{H}_{ac}(F)$.

Let us prove that the time–dependent formulae (1.24) hold. In fact, we shall prove a more general result known as the invariance principle [29]. Let $f(\lambda)$ be a real–valued measurable function defined on $\mathbb{R}$ such that,

$$\lim_{\tau\to\infty}\int_0^{\infty}\left|\int_{\mathbb{R}}\varphi(\lambda)e^{-i\tau f(\lambda)-ix\lambda}d\lambda\right|^2 dx = 0, \tag{4.70}$$

for all $\varphi \in L^2(\mathbb{R})$. Then,

$$\mathcal{W}_{\pm} = \mathrm{s}-\lim_{\tau\to\pm\infty} e^{i\tau f(F)}e^{-i\tau f(F_0)}. \tag{4.71}$$

Note that $f(\lambda)=\lambda$ satisfies (4.70) and then, (1.24) hold. We give some details of the proof in the $+$ case. By unitarity it is enough to prove that

$$\lim_{\tau\to\infty}(e^{-i\tau f(F)}E(I_1)A\varphi, e^{-i\tau f(F_0)}E_0(I_0)A\psi) = (E(I_1)A\varphi, \mathcal{F}_{+}^{*}\mathcal{F}_0E_0(I_0)A\psi), \tag{4.72}$$

for $\varphi,\psi \in D((1+|x|)^{(1+\varepsilon)/2}A)$ and I_0, I_1 bounded intervals with $\bar{I}_0, \bar{I}_1 \subset \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F)$. But by (2.25), (2.39), (4.34), (4.37) and (4.67),

$$\begin{aligned}(e^{-i\tau f(F)}E(I_1)A\varphi,\ e^{-i\tau f(F_0)}E_0(I_0)A\psi) = (E(I_1)A\varphi, \mathcal{F}_{+}^{*}\mathcal{F}_0E_0(I_0)A\psi)+\\ \int_{I_1} e^{-i\tau f(\lambda)}\lim_{\varepsilon\downarrow 0}h_{\varepsilon,\tau}(\lambda)d\lambda,\end{aligned} \tag{4.73}$$

where

$$h_{\varepsilon,\tau}(\lambda) := \frac{1}{2\pi i}([G_{+}(\lambda+i\varepsilon)-G_{-}(\lambda-i\varepsilon)]\varphi, AR_{0,+}(\lambda+i\varepsilon)e^{-i\tau f(F_0)}E_0(I_0)A\psi). \tag{4.74}$$

We prove that

$$\lim_{\tau\to\infty}\int_{I_1} e^{-i\tau f(\lambda)}\lim_{\varepsilon\downarrow 0}h_{\varepsilon,\tau}(\lambda)d\lambda = 0, \tag{4.75}$$

as in the proof of Theorem 3, Section 6, Chapter 5 of [23] (see also the proof of Lemma 7.3 of [35]). By (the proof of) Theorem 4 and Corollary 1 of [9] the wave operators $W_{\pm}(\tau)$ exist and (1.25) hold.

For this purpose note that if $\varphi \in D(H_0), U_0(t,0)\varphi$ is globally Lipschitz continuous. Furthermore, by the proof of Theorem 3.3 given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\|(U(t_2,t_1) - I)\phi\|_{L^2} \leq \varepsilon, \text{ if } |t_2 - t_1| \leq \delta, \tag{4.76}$$

and where δ depends only on the L^2 norm of ϕ. Then, $(U(t,0))^{-1}U_0(t,0)\varphi$ is uniformly continuous if $\varphi \in D(H_0)$. We prove that Range $W_\pm(t) = \mathcal{H}_{ac}(U(t+2\pi,t))$ arguing as in the proof of Theorem 1.1 of [43].

■

By the intertwining relations [29],

$$F P_{ac}(F) = \mathcal{W}_\pm \, F_0 \, \mathcal{W}_\pm^*, \tag{4.77}$$

and, in particular, the absolutely–continuous spectrum of F coincides with the spectrum of F_0, and it is equal to $\mathbb{R}$.

As in the proof of Theorem 6.3 of [22] we prove that (see (1.17)) for $\lambda \in \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F)$

$$\hat{\mathcal{S}}(\lambda) = I - 2\pi i D(\lambda) A G_+(\lambda) B D^*(\lambda), \tag{4.78}$$

$$\hat{\mathcal{S}}(\lambda)^{-1} = I + 2\pi i D(\lambda) A G_-(\lambda) B D^*(\lambda). \tag{4.79}$$

$\hat{\mathcal{S}}(\lambda)$ is unitary on $\hat{\mathcal{H}}(\lambda)$. Note that by (2.39) and Lemma 4.1 $D(\lambda)A$ and $D(\lambda)B$ are compact operators, and hence, $\hat{\mathcal{S}}(\lambda) - I$ is compact. Suppose that in (4.1) $s := \frac{1+\varepsilon}{2} > \frac{3}{2}$ and define,

$$\psi_m := V_2 \, \phi_m \tag{4.80}$$

with ϕ_m as in (2.13). Since $\phi_m \in \mathcal{K}_{1,-s}, \psi_m \in \mathcal{H}$. Let us denote

$$\psi_{-,m}(t,x,\lambda,\nu) = \psi_m(t,x,\lambda,\nu) - V_2 R_+(\lambda) V_1 \, \psi_m(\cdot,\cdot,\lambda,\nu), \tag{4.81}$$

$m < \lambda, \lambda \in \mathbb{R}\backslash\mathbb{Z}\backslash\sigma_p(F)$, $\nu \in S_1^2$. By taking adjoint in (4.32) and by (4.37) and (4.81) we prove that $\psi_{-,m}$ is a solution of the following Lippmann-Schwinger equation

$$\psi_{-,m} = \psi_m - V_2 R_{0,+}(\lambda) V_1 \psi_{-,m}. \tag{4.82}$$

Recall that by Lemma 4.1 $V_2 R_{0,+}(\lambda) V_1$ is compact in $\mathcal{H}$, and then (4.82) has an unique solution unless there is a non-trivial solution to the homogeneous equation

$$\varphi = -V_2 R_{0,+}(\lambda) V_1 \varphi. \tag{4.83}$$

But as $V_1\varphi = \chi_1 V_1 \varphi$ (4.83) is equivalent to

$$\chi_1 \varphi = -Q_{0,+}(\lambda)\chi_1\varphi, \tag{4.84}$$

and as by (4.83) $\chi_1\varphi \not\equiv 0$, by Lemma 4.2 equation (4.82) has an unique solution for $\lambda \in \mathbb{R}\backslash\mathbb{Z}$ if and only if $\lambda \notin \sigma_p(F)$. Define,

$$\phi_{-,m}(t,x,\lambda,\nu) = \phi_m(t,x,\lambda,\nu) - R_{0,+}(\lambda) V_1 \psi_{-,m}(\cdot,\cdot,\lambda,\nu). \tag{4.85}$$

By (4.85), and as $\psi_{-,m} = V_2\phi_{-,m}$,

$$(F_0 + V)\phi_{-,m} = \lambda \phi_{-,m}. \tag{4.86}$$

Observe that $(F_0 - \lambda)\phi_{-,m}$ and $V\phi_{-,m}$ belong to $\mathcal{K}_{-1,s}$ and that (4.86) holds on $\mathcal{K}_{-1,-s}$.

Let $T(\lambda) := \hat{\mathcal{S}}(\lambda) - I$ be the scattering amplitude. Then, by (2.13), (2.21), (2.22), (4.36) and (4.78), for $\varphi \in \hat{\mathcal{H}}(\lambda)$

$$(T(\lambda)\varphi)_n(\nu) = -2\pi i \sum_{m<\lambda} \int_{S_1^2} T_{n,m}(\lambda,\nu,\nu')\varphi_m(\nu')\, d\nu', \tag{4.87}$$

where

$$T_{n,m}(\lambda,\nu,\nu') := \left(V\phi_{-,m}(\cdot,\cdot,\lambda,\nu'), \phi_n(\cdot,\cdot,\lambda,\nu)\right). \tag{4.88}$$

Our results on the existence and completeness of the wave operators extend those of [10], [43], to the critical singularity $L^{3/2}$ of the potential. For other results on scattering with time-periodic potentials see for example [3], [21], [25], [31], [42] and [44].

5 The Inversion

We first prepare some results on the Faddeev's Green operator that we need. For the proofs see [36].

1) For $p = p_\perp + z\nu \in \mathbb{C}^3, p_\perp \in \mathbb{R}^3, z \in \mathbb{C}_\pm, \nu \in S_1^2, p_\perp \cdot \nu = 0$, denote,

$$g_\nu(p)\varphi := \frac{1}{(2\pi)^{3/2}} \int e^{ik\cdot x} \frac{1}{(k^2 + 2p\cdot k)} \hat{\varphi}(k) dk \tag{5.1}$$

where $\hat{\varphi} = \mathcal{F}_T \varphi$. Then, as a function of $p_\perp, \nu, z \in \mathbb{C}_\pm$, $g_\nu(p)$ is continuous and it has continuous extensions to $z \in \overline{\mathbb{C}}_\pm$ with values in $\mathcal{B}(L^2_s, W_{2,-s}), s > 1/2$, with the exception of $(p_\perp, z) = (0,0)$, and if $s > 1$ also at $(p_\perp, z) = (0,0)$.

2) For fixed $p_\perp, \nu, g_\nu(p_\perp, z)$ is an analytic function of $z \in \mathbb{C}_\pm$ with values in $\mathcal{B}(L^2_s, W_{2,-s})$.

3) For any $\delta > 0$ there is a constant C_δ such that,

$$\|g_\nu(p)\|_{\mathcal{B}(L^2_s, W_{\rho,-s})} \leq C_\delta(|p_\perp| + |z|)^{\rho-1}, 0 \leq \rho \leq 2, \tag{5.2}$$

for $|p_\perp| + |z| \geq \delta$, $s > 1/2$.

4) For $p_\nu \in \mathbb{R}$,

$$\begin{gathered} g_{\nu,\pm}(p_\perp, p_\nu) := \mathrm{s} - \lim_{\epsilon \downarrow 0} g_\nu(p_\perp + (p_\nu \pm i\epsilon)\nu) = \\ e^{-i(p_\perp + p_\nu \nu)\cdot x} \left(R_{0,+}((p_\perp + p_\nu\nu)^2) - \frac{i}{8\pi^2 |p_\perp + p_\nu \nu|} T^*(|p_\perp + p_\nu\nu|) \chi_{\{\pm\omega_\nu > \pm p_\nu/|p_\perp + p_\nu\nu|\}} \right. \\ \left. T(|p_\perp + p_\nu\nu|) \right) e^{i(p_\perp + p_\nu\nu)\cdot x}. \end{gathered} \tag{5.3}$$

For (5.2) with $p^2 = 0, p \neq 0$, see [33].

Consider now the operator,

$$g_\nu(p,\gamma)\varphi := \frac{1}{(2\pi)^{3/2}} \int \frac{e^{ik\cdot x}}{k^2 + 2p\cdot k - \gamma} \hat{\varphi}(k) dk, \tag{5.4}$$

with $p = p_\perp + z\nu, z = \alpha + i\beta \in \mathbb{C}_\pm$, $\gamma \in \mathbb{R}$. If $\gamma \geq 0$,

$$g_\nu(p,\gamma) = e^{i\sqrt{\gamma}\,\omega\cdot x} g_\nu(p_\perp + z\nu + \sqrt{\gamma}\omega) e^{-i\sqrt{\gamma}\,\omega\cdot x}, \tag{5.5}$$

where $\omega \in S_1^2$ satisfies, $\omega\cdot p_\perp = \omega\cdot\nu = 0$. Then, by (5.2) for any $\delta > 0$,

$$\|g_\nu(p,\gamma)\|_{\mathcal{B}(L^2_s, W_{\rho,-s})} \leq C_\delta \left(p_\perp^2 + \alpha^2 + \beta^2 + \gamma\right)^{(\rho-1)/2}, 0 \leq \rho \leq 2, s > 1/2, \tag{5.6}$$

for $(p_\perp^2 + \alpha^2 + \beta^2 + \gamma)^{1/2} \geq \delta$. In the case where $\gamma < 0$, we prove as in the proof of Theorem 1.1 and Remark 2.2 of [36] that for any $\delta > 0$ there is a constant C_δ such that,

$$\|g_\nu(p,\gamma)\|_{\mathcal{B}(L^2_s, W_{\rho,-s})} \leq C_\delta \frac{(1 + (p_\perp^2 + \alpha^2 + \beta^2 + |\gamma|)\,(|p_\perp^2 + \alpha^2 + \gamma| + \beta^2)^{-1/2})^{\rho/2}}{(|p_\perp^2 + \alpha^2 + \gamma| + \beta^2)^{(1-\rho/2)/2}}, 0 \leq \rho \leq 2, \tag{5.7}$$

$s > 1/2$, for $(|p_\perp^2 + \alpha^2 + \gamma| + \beta^2)^{1/2} \geq \delta$.

We now proceed as in [34]. For $\delta \in \mathbb{R}$ denote,

$$\mathcal{E}_\delta := \{\varphi : e^{\delta|x|}\varphi(x) \in L^2\}, \tag{5.8}$$

with norm

$$\|\varphi\|_{\mathcal{E}_\delta} := \|e^{\delta|x|}\varphi(x)\|_{L^2}, \tag{5.9}$$

and

$$\mathcal{E}_{1,\delta} := \{\varphi \in \mathcal{E}_\delta : \frac{\partial}{\partial x_j}\varphi \in \mathcal{E}_\delta, j = 1,2,3\}, \tag{5.10}$$

with norm

$$\|\varphi\|_{\mathcal{E}_{1,\delta}} := \left[\|\varphi\|^2_{\mathcal{E}_\delta} + \sum_{j=1}^{3} \left\|\frac{\partial}{\partial x_j}\varphi\right\|^2_{\mathcal{E}_\delta}\right]^{1/2}. \tag{5.11}$$

Suppose that $\delta > 0$ and fix $\nu \in S_1^2$. Then, if $\gamma > 0$ there is a neighborhood, O, of $\mathbb{R}^2 \times \mathbb{C}\setminus\mathbb{R}$, in $\mathbb{C}^2 \times \mathbb{C}\setminus\mathbb{R}$, and if $\gamma \leq 0$ there is a neighborhood, O, of $(\mathbb{R}^2 \setminus S^2_{\sqrt{|\gamma|}}) \times \mathbb{C}\setminus\mathbb{R}$, in $\mathbb{C}^2 \times \mathbb{C}\setminus\mathbb{R}$, such that for any $(p_\perp, z) \in O$ there is an operator $h_\nu(p_\perp, z, \gamma) \in \mathcal{B}(\mathcal{E}_\delta, \mathcal{E}_{-\delta})$ that is analytic in O and such that

$$h_\nu(p_\perp, z, \gamma) = g_\nu(p_\perp + z\nu, \gamma), p_\perp \in \mathbb{R}^2, z \in \mathbb{C}\setminus\mathbb{R}. \tag{5.12}$$

Moreover, for fixed $p_\perp, z, \nu$ the family of operators $h_\nu(p_\perp, z, \gamma)$ is uniformly bounded for γ such that, $|p_{\perp,R}^2 + \gamma| \geq \eta$, for any $\eta > 0$, where $p_{\perp,R}$ denotes the real part of $p_\perp$.

Designate by $-\Delta_\perp$ the Laplacian on the plane orthogonal to ν and,

$$r_{\perp,\pm}(z) := (-\Delta_\perp - z)^{-1}, z \in \overline{\mathbb{C}_\pm} \backslash \{0\}. \tag{5.13}$$

The $r_{\perp,\pm}(z)$ are integral operators with integral kernel $\frac{i}{4} H_0^{(1)}(\sqrt{z}|x-y|)$, with $H_0^{(1)}$ the modified Hankel function. As is well known, this implies that $r_{\perp,\pm}(z)$ have analytic continuations across $(0,\infty)$ to $|\Im\sqrt{z}| < \delta$ as operators on $\mathcal{B}(\mathcal{E}_\delta, \mathcal{E}_{-\delta})$ for any $\delta > 0$, with bound uniform for $|z| \geq \eta_1, |\Im z| \leq \delta - \eta_2$, for any $\eta_1, \eta_2 > 0$. Let $f \in C_0^\infty(\mathbb{R})$ satisfy, $f(\xi) = 1, |\xi| \leq \varepsilon, f(\xi) = 0, |\xi| \geq 2\varepsilon$ with ε small enough. Then, $h_\nu(p_\perp, z, \gamma)$ is defined as

$$h_\nu(p_\perp, z, \gamma) := h_\nu^{(1)}(p_\perp, z, \gamma) + h_\nu^{(2)}(p_\perp, z, \gamma), \tag{5.14}$$

where,

$$h_\nu^{(1)}(p_\perp, z, \gamma) := e^{-ip_\perp \cdot x} \mathcal{F}_T^{-1} \left(\frac{(1 - f(k_\nu))}{k^2 + 2zk_\nu - \gamma - p_\perp^2} \right) \mathcal{F}_T \, e^{ip_\perp \cdot x}, \tag{5.15}$$

and,

$$\begin{aligned} h_\nu^{(2)}(p_\perp, z, \gamma) := e^{-ip_\perp \cdot x} \mathcal{F}_\nu^{-1} [& r_{\perp,+}(-k_\nu^2 - 2zk_\nu + \gamma + p_\perp^2) \chi_{(-\infty,0)}(k_\nu) \\ & + r_{\perp,-}(-k_\nu^2 - 2zk_\nu + \gamma + p_\perp^2) \chi_{(0,\infty)}(k_\nu)] f(k_\nu) \mathcal{F}_\nu \, e^{ip_\perp \cdot x}, \end{aligned} \tag{5.16}$$

where $\mathcal{F}_\nu$ is the Fourier transform along the ν direction in $\mathbb{R}^3$ and $k_\nu = k \cdot \nu$.

In [34] the case $\gamma = 0$ was considered.

Recall that $f_1, f_2 \in L^3$ are compact operators from $W_{1,2}$ into L^2. Then, $Q_\nu(p_\perp, z, \gamma) := e^{-\delta_0|x|} f_1 h_\nu^{(1)}$ $e^{-\delta_0|x|} f_2$ is compact in L^2, and its norm tends to zero as $\gamma \to -\infty$. In the case $\gamma > 0$ we write, $z\nu + p_\perp = q + z_1\mu$, with $q \in \mathbb{R}^3, z_1 \in \mathbb{C} \setminus \mathbb{R}, \mu \in S_1^2, q \cdot \mu = 0$ and $\Im z_1 > 0$. Then,

$$Q_\nu(p_\perp, z, \gamma) = f_1(x) \, e^{i\sqrt{\gamma}\omega \cdot x} e^{-\delta_0|x|} g_\mu(p) \mathcal{F}_\nu^{-1} (1 - f(k_\nu)) \mathcal{F}_\nu e^{-\delta_0|x|} \, e^{-i\sqrt{\gamma}\omega \cdot x} f_2(x), \tag{5.17}$$

where, $p := q + z_1\mu + \sqrt{\gamma}\omega$ with $\omega \in S_1^2, q \cdot \omega = \mu \cdot \omega = 0$. $\mathcal{F}_\nu^{-1}(1 - f(k_\nu))\mathcal{F}_\nu$ is bounded in L_s^2. Moreover, for ϕ, ψ in Schwartz space, $(Q_\nu(p_\perp, z, \gamma)\phi, \psi)$ is analytic in z_1 and it tends to zero as

$\Im z_1 \to \infty$. Hence, it follows from the maximum-modulus principle (see [2], page 231) that it takes the maximum for z_1 real.

Moreover (see the proof of Lemma 4.1), $f_1 r_{0,\pm}(z) f_1$ is bounded in L^2 with bound uniform for $z \in \overline{\mathbb{C}_\pm}$. Denote, $d(\rho) := (1/\sqrt{2}\rho^{1/4})\, T(\sqrt{\rho})$ with $T(\rho)$ the trace operator (2.16). As,

$$f_1\, d^*(\rho)\, d(\rho) f_1 := \frac{1}{2\pi i} f_1 \left[r_{0,+}(\rho) - r_{0,-}(\rho) \right] f_1, \tag{5.18}$$

we have that $d(\rho) f_1 \in \mathcal{B}(L^2, L^2(S_1^2))$ with bound uniform in ρ. Moreover, if $f_1 \in C_0^\infty$, $f_1\, r_{0,\pm}(\rho)\, f_1$ is Hilbert-Schmidt, and hence, in this case, $d(\rho) f_1$ is compact. Approximating f_1 by functions in C_0^∞ we prove that $d(\rho) f_1$ is compact for $f_1 \in L^3$.

Furthermore, for z_1 real, $\Im z = \Im p_\perp = 0$ and it follows from (5.3) that in this case,

$$|(Q_\nu(p_\perp, z, \gamma)\phi, \psi)| \leq C \|f_1\|_{L^3}\, \|f_2\|_{L^3}\, \|\phi\|_{L^2} \|\psi\|_{L^2}, \tag{5.19}$$

uniformly in $\gamma > 0$. Here we use that $\mathcal{F}_\nu^{-1}(1 - f(k_\nu))\mathcal{F}_\nu$ is bounded in all L^p spaces. By the maximum-modulus argument above, this is also true for all $z \in \mathbb{C}$, and by continuity, it also holds for all $\phi, \psi \in L^2$, and it follows that $Q_\nu(p_\perp, z, \gamma)$ is uniformly bounded in L^2 for all $\gamma > 0$. Note that if $f_1, f_2 \in C_0^\infty$, the norm of $Q_\nu(p_\perp, z, \gamma)$ goes to zero as $\gamma \to \infty$. Hence, approximating f_1, f_2 by functions in C_0^∞ we prove that this is also true for $f_1, f_2 \in L^3$.

The function $H_0^{(1)}(z)$ satisfies the following estimate [7],

$$|H_0^{(1)}(z)| \leq C \begin{cases} |\ln z|, & |z| \leq 1/2, \\ \frac{e^{|\Im z|}}{|z|^{1/2}}, & |z| \geq 1/2. \end{cases} \tag{5.20}$$

From this it follows that $e^{-\delta_0 |x|}\, f_1\, h_\nu^{(2)}\, e^{-\delta_0|x|} f_2$ is Hilbert-Schmidt in L^2 and that its norm goes to zero as $|\gamma| \to \infty$.

For any $\delta \in \mathbb{R}$, we define,

$$\mathcal{D}_\delta := L^2(T, \mathcal{E}_\delta). \tag{5.21}$$

For

$$(p_\perp, z) \in O_{\mathbb{Z}} := \left\{ (p_\perp, z) \in O, \text{ with } p_{\perp,R}^2 \notin \mathbb{Z} \right\}, \tag{5.22}$$

let us define,

$$H_\nu(p_\perp, z, m) := \oplus_{n=-\infty}^{\infty} h_\nu(p_\perp, z, m-n)\, P_n \in \mathcal{B}(\mathcal{D}_\delta, \mathcal{D}_{-\delta}). \tag{5.23}$$

If V satisfies the conditions of Theorem 1.1, we prove as in Lemma 4.1- using the results above- that

$$M_\nu(p_\perp, z, m) := V_2 H_\nu(p_\perp, z, m) V_1, (p_\perp, z) \in O_{\mathbb{Z}}, \tag{5.24}$$

are compact operators on $\mathcal{H}$ and, moreover, that $M_\nu(p_\perp, z, m)$ go to zero in norm as $p_\perp \in \mathbb{R}^2, |\Im z| \to \infty$, uniformly in $p_\perp$. Then, $(I + M_\nu(p_\perp, z, m))$ has a bounded inverse in $\mathcal{H}$ for $(p_\perp, z) \in O \backslash O_e(m)$, where the exceptional set $O_e(m)$ has the following properties. The intersection of $O_e(m)$ with $\{(p_\perp, z) \in O_{\mathbb{Z}} : p_\perp \in \mathbb{R}^3, |\Im z| \geq M\}$ is empty for some $M > 0$. Let $U \subset \mathbb{C}_\pm$ be any open set such that there is a sequence, $z_n \in U$ with $|\Im z_n| \to \infty$ as $n \to \infty$. Moreover, let $z \in U, \to p_\perp(z)$ be any analytic function such that $(p_\perp(z), z) \in O_{\mathbb{Z}}, z \in U$ and that $p_\perp(z_n) \in \mathbb{R}^3$. Then, the set $\{z \in U : (p_\perp(z), z) \in O_e(m)\}$ has no accumulation points in U.

Let us now define for $\lambda \in \mathbb{R} \backslash \mathbb{Z} \backslash \sigma_p(F), \psi_{+,m}$ as in (4.81), but with $R_-(\lambda)$ instead of $R_+(\lambda)$,

$$\psi_{+,m}(t, x, \lambda, \nu) := \psi_m(t, x, \lambda, \nu) - V_2 R_-(\lambda) V_1 \psi_m(\cdot, \cdot, \lambda, \nu), \tag{5.25}$$

and (see (4.85))

$$\phi_{+,m}(t, x, \lambda, \nu) := \phi_m(t, x, \lambda, \nu) - R_{0,-}(\lambda) V_1 \psi_{+,m}(\cdot, \cdot, \lambda, \nu). \tag{5.26}$$

We define for $g \in \hat{\mathcal{H}}(\lambda)$,

$$\phi_{\pm,g}(t, x, \lambda) := \sum_{m<\lambda} \int_{S_1^2} \phi_{\pm,m}(t, x, \lambda, \nu) g_m(\nu) d\nu. \tag{5.27}$$

Suppose that we are given another potential, $\tilde{V}$, that satisfies the conditions of Theorem 1.1, and let us denote by $\psi^{\sim}_{\pm,m}, \phi^{\sim}_{\pm,m}, S^{\sim}(\lambda), \tilde{T}_{n,m}(\lambda)$ the corresponding quantities for $\lambda \in \mathbb{R} \backslash \mathbb{Z} \backslash \sigma_p(\tilde{F})$. It follows from (4.81), (4.85), (5.25), and (5.26) that,

$$\begin{aligned} 2\pi i \Big[((F_0 - \lambda)\phi_{-,n}, \tilde{\phi}_{+,m}) \quad &- \quad (\phi_{-,n}, (F_0 - \lambda)\tilde{\phi}_{+,m})\Big] \\ = -2\pi i \Big[(V\phi_{-,n}, \phi_m) \quad &- \quad (\tilde{V}\tilde{\phi}_{-,n}, \phi_m)\Big]. \end{aligned} \tag{5.28}$$

$$\tag{5.29}$$

Then, by (4.87) and (4.88)

$$2\pi i\Big[((F_0-\lambda)\phi_{-,g},\tilde{\phi}_{+,\tilde{g}})-(\phi_{-,g},(F_0-\lambda)\tilde{\phi}_{+,\tilde{g}})\Big]=\Big((S(\lambda)-\tilde{S}(\lambda))g,\tilde{g}\Big)\,,g,\tilde{g}\in\hat{\mathcal{H}}(\lambda). \tag{5.30}$$

Moreover, for $(p_\perp,z)\in O\backslash O_e(m)$ let us denote,

$$\Omega_{m,\nu}(t,x,p_\perp,z):=e^{i(p_\perp+z\nu)\cdot x}[e^{imt}-H_\nu(p_\perp,z,m)V_1\Gamma_{m,\nu}(\cdot,\cdot,p_\perp,z)], \tag{5.31}$$

where,

$$\Gamma_{m,\nu}(t,x,p_\perp,z):=(1+V_2H_\nu(p_\perp,z,m)V_1)^{-1}e^{imt}V_2, \tag{5.32}$$

is the unique solution of the Lippmann-Schwinger equation,

$$\Gamma_{m,\nu}(t,x,p_\perp,z)=e^{imt}V_2-V_2H_\nu(p_\perp,z,m)V_1\Gamma_{m,\nu}(\cdot,\cdot,p_\perp,z) \tag{5.33}$$

in $\mathcal{H}$. As $e^{-\delta|x|}H_\nu(p_\perp,z,m)V_1\in\mathcal{B}(\mathcal{H})$ for $\delta>0$ we have that,

$$e^{-\delta|x|}H_\nu(p_\perp,z,m)\,V_1\,\Gamma_{m,\nu}(\cdot,\cdot,p_\perp,z)\in\mathcal{H}. \tag{5.34}$$

Then, $\Omega_{m,\nu}\in\mathcal{D}_{-\delta}$, if $\delta>|\Im z|+|\Im p_\perp|$. Moreover, if $(p_\perp+z\nu)^2=\lambda-m$, $(F_0-\lambda)\,\Omega_{m,\nu}\in\mathcal{D}_{-1,\delta_0}, V_2\,\Omega_{m,\nu}\in\mathcal{D}_{\delta_0}$, and then, $V\,\Omega_{m,\nu}=V_1V_2\,\Omega_{m,\nu}\in\mathcal{D}_{-1,2\delta_0}$. Hence,

$$(F_0+V-\lambda)\Omega_{m,\nu}=0, \tag{5.35}$$

in $\mathcal{D}_{-1,-\delta}$.

By (4.34) and using (4.37), (4.81), (4.85), (5.25) and (5.26), we see that

$$\phi_{\pm,m}(t,x,\lambda,\nu)=\phi_m(t,x,\lambda,\nu)-R_\mp(\lambda)V_1\psi_m(\cdot,\cdot,\lambda,\nu). \tag{5.36}$$

Let Q be any function that satisfies the conditions for $V_j, j=1,2$ in Theorem 1.1. Let $f\in\mathcal{H}$, and suppose that

$$(f,Q\phi_{+,g})=0, \tag{5.37}$$

for all $g\in\hat{\mathcal{H}}(\lambda)$. Denote,

$$h:=(I-AG_-(\lambda)BR_{0,-}(\lambda))Qf. \tag{5.38}$$

Then (see (2.13) and (2.21)), for $\lambda > m, g_m \in L^2(S_1^2)$,

$$(T_m(\lambda)h, g_m) = \int_{S_1^2} \bar{g}_m(\nu)(f, Q\,\phi_{+,m}(\cdot,\cdot,\lambda,\nu))d\nu = 0, \tag{5.39}$$

where we have taken adjoints in (4.32) and we used (4.36), and (5.36). It follows that if (5.37) holds then (see (2.22)),

$$D(\lambda)h = 0. \tag{5.40}$$

Let us designate,

$$u_- := R_-(\lambda)Qf. \tag{5.41}$$

By (4.33) and (4.37),

$$u_- = R_{0,-}(\lambda)h \in \mathcal{H}_{-s}, s > 1/2. \tag{5.42}$$

Then, by (5.40) it follows as in the proof of Lemma 4.2 that $u_- \in \mathcal{H}$. Note that $h \in \mathcal{E}_{-1,\delta}, \delta < \delta_0$. By (5.42), as $u_- \in \mathcal{H}$ and (5.40) holds,

$$(\tilde{\mathcal{F}}u_-)_m(k) = \frac{1}{k^2 - \lambda + m}(\tilde{\mathcal{F}}h)_m(k), \tag{5.43}$$

with $(\tilde{\mathcal{F}}h)_m(\sqrt{\lambda - m}\,\nu) = 0, \nu \in S_1^2, \lambda > m$. It follows from the Paley–Wiener theorem [27] that $u_- \in \mathcal{E}_{1,\delta}$, for any $\delta < \delta_0$.

Arguing as in the beginning of the proof of Lemma 4.2 we obtain that,

$$(F_0 + V - \lambda)u_- = Qf. \tag{5.44}$$

For $\delta < \delta_0$ denote,

$$N := \{\varphi \in \mathcal{H} : \varphi = Q\phi, \phi \in \mathcal{E}_{-\delta}, (F_0 - \lambda)\phi \in \mathcal{D}_{-1,-\delta}, V\phi \in \mathcal{D}_{-1,-\delta}, \text{ and } (F_0 + V - \lambda)\phi = 0\}. \tag{5.45}$$

Then, for any $\varphi \in N$:

$$(f, \varphi) = ((F_0 + V - \lambda)u_-, \phi) = 0. \tag{5.46}$$

This implies that $f \in N^{\perp}$, and hence, the set

$$\left\{ \varphi : \varphi = Q\phi_{+,g}, \text{ with } g \in \hat{\mathcal{H}}(\lambda) \right\} \tag{5.47}$$

is dense in the closure, $\overline{N}$, of N in $\mathcal{H}$. We prove that the set

$$\left\{ \varphi : \varphi = Q\phi_{-,g}, \text{ with } g \in \hat{\mathcal{H}}(\lambda) \right\} \tag{5.48}$$

is dense in $\overline{N}$ in the same way.

We argue now as in [34]. For any $m_1, m_2 \in \mathbb{Z}$ denote $m_0 = \max[m_1, m_2]$. Suppose that $\lambda > m_0$. Then, for any $k \in \mathbb{R}^3$ take $\nu, \omega \in S_1^2$ with $k \cdot \nu = k \cdot \omega = \nu \cdot \omega = 0$, and for $\rho > 0$ such that $\rho^2 + \lambda - m_j \notin \mathbb{Z}, j = 1, 2$, define

$$p_{\perp} := \frac{k}{2} + (\rho^2 - \frac{1}{4}k^2 + \lambda - m_1)^{1/2}\omega, \tag{5.49}$$

$$p'_{\perp} := -\frac{k}{2} + (\rho^2 - \frac{1}{4}k^2 + \lambda - m_2)^{1/2}\omega, \tag{5.50}$$

for $\rho > \left[\max(\frac{1}{4}k^2 - \lambda + m_0, 0)\right]^{1/2}$. The integral,

$$I := \int (V - \tilde{V})(t, x)\Omega_{m_1,\nu}(t, x, p_{\perp}, i\rho)\overline{\tilde{\Omega}_{m_2,\nu}(t, x, p'_{\perp}, -i\rho)}\, dt\ dx \tag{5.51}$$

converges and it is meromorphic for ρ in a neighborhood of $\rho > \left[\max(\frac{1}{4}k^2 - \lambda + m_0, 0)\right]^{1/2}$. Moreover, by (5.30) and the density argument above, $I = 0$ for $\rho < \delta_0$. If for some ϵ with, $\delta_0 > \epsilon > 0$, $k^2 < 4\left[(\delta_0 - \epsilon)^2 + \lambda - m_0\right]$ this contains all the ρ as above with $\delta_0 - \epsilon < \rho < \delta_0$. It follows by analyticity in ρ that for these k, $I = 0$ for $\rho > \left[\max(\frac{1}{4}k^2 - \lambda + m_0, 0)\right]^{1/2}$. Taking the limit $\rho \to \infty$ we have that,

$$\int e^{ik\cdot x} e^{i(m_1 - m_2)t}[V(t, x) - \tilde{V}(t, x)]dt\ dx = 0. \tag{5.52}$$

But as the Fourier transform of $V - \tilde{V}$ is analytic in k for $|\Im k| < 2\delta_0$, this holds for all $k \in \mathbb{R}^3, m_1, m_2 < \lambda$. This implies that $V - \tilde{V} \equiv 0$, and it completes the proof of Theorem 1.1.

References

[1] Agmon, S.: Spectral properties of Schrödinger operators and scattering theory. Ann. Scuola Norm. Sup. Pisa Cl Sci. **2**, 151–218 (1975).

[2] Dunford, N.; Schwartz, J.T.:*Linear Operators, Part I: General Theory.* New York: Interscience, 1964.

[3] Enss, V., Veselić, K.: Bound states and propagating states for time–dependent Hamiltonians. Ann. Inst. H. Poincaré Phys. Théor. XXXIX, 159–191 (1983).

[4] Eskin, G., Ralston, J.: Inverse scattering problem for the Schrödinger equation with magnetic potential at a fixed energy. Comm. Math. Phys. **173**, 199–224 (1995).

[5] Faddeev, L. D.: Inverse problems of quantum scattering theory, II. Itogi Nauki i Tekniki Sovrem. Prob. Mat. **3**, 93–180 (1974) [Engl. transl. J. Sov. Mat. **5**, 334–396 (1976)].

[6] Galtbayar, A., Jensen, A., Yajima, K.: Local time–decay of solutions to Schrödinger equations with time–periodic potentials. Preprint 2002.

[7] Gradstein, I.S., Ryzhik, I.M.: *Table of Integrals Series and Products.* New York: Academic Press, 1965.

[8] Guillot, J.C., Ralston, J.: Inverse Scattering at fixed energy for layered media. J. Math. Pures Appl. **78**, 27–48 (1999).

[9] Howland, J.: Stationary scattering theory for time–dependent Hamiltonians. Math. Ann. **207**, 315–335 (1974).

[10] Howland, J.: Scattering theory for Hamiltonians periodic in time. Indiana Univ. Math. J. **28**, 471–494 (1974).

[11] Howland, J.: Two problems with time–dependent Hamiltonians. In: De Santo, J.A., Sáenz, A.W. and Zachary, W.W. (eds.), *Mathematical Methods and Applications of Scattering Theory*, Lecture Notes in Physics, **130**, Berlin: Springer, 1980, pp 163–168.

[12] Isozaki, H.: Inverse scattering theory for Dirac operators. Ann. Inst. H. Poincaré, Phys. Theor. **66**, 237–270 (1997).

[13] Isozaki, H.: Inverse scattering theory for wave equations in stratified media. J. Differential Equations **138**, 19–54 (1997).

[14] Jensen, A., Kato, T.: Spectral properties of Schrödinger operators and time decay of the wave function. Duke Math. J. **46**, 583–611 (1979).

[15] Jensen, A., Mourre, E., Perry, P.: Multiple Commutator estimates and resolvent smoothness in quantum scattering theory. Ann. Inst. H. Poincaré Phys. Théor. **41**, 207–225 (1984).

[16] Kato, T.: Integration of the equations of evolution in a Banach space. J. Math. Soc. Japan **5**, 200–234 (1953).

[17] Kato, T.: Wave operators and similarity for some non–selfadjoint operators. Math. Ann. **162**, 258–279 (1966).

[18] Kato, T.: Linear evolution equations of hyperbolic type I. J. Fac. Sci. Univ. Tokyo Sect I A Math. **17**, 241–258 (1970).

[19] Kato, T., Kuroda, S.T.: Theory of simple scattering and eigenfunction expansions. In: *Functional Analysis and related fields*, Berlin: Springer, 1970, pp 99–131 .

[20] Keel, M., Tao, T.: Endpoint Strichartz estimate. Amer. J. Math. **120**, 955-980 (1998).

[21] Kitada, H., Yajima, K.: A scattering theory for time–dependent long–range potentials. Duke Math. J. **49**, 341–376 (1982).

[22] Kuroda, S.T.: Scattering theory for differential operators I. J. Math. Soc. Japan **25**, 74–104 (1973).

[23] Kuroda, S.T.: *An Introduction to Scattering Theory*. Aarhus: Lecture Notes Series No. 51, Aarhus Universitet, 1980.

[24] Kuwabara, Y., Yajima, K.: The limiting absorption principle for Schrödinger operators with long–range time–periodic potentials. J. Fac. Sci. Univ. Tokyo. Sect. IA, Math. **34**, 833–851 (1987).

[25] Martin, Ph. A., Sassolide Bianchi, M.: On the low–and high–frequency limit of quantum scattering by time–dependent potentials. J. Phys. A. Math. Gen. **28**, 2403–2427 (1995).

[26] Novikov, R.G.: The inverse scattering problem at fixed energy for the three dimensional Schrödinger equation with an exponentially decreasing potential. Commun. Math. Phys. **161**, 569–595 (1994).

[27] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics II. Fourier Analysis Self–Adjointness*. New York: Academic Press, 1975.

[28] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics IV Analysis of Operators*. New York: Academic Press, 1978.

[29] Reed, M., Simon, B.: *Methods of Modern Mathematical Physics III Scattering Theory*. New York: Academic Press, 1979.

[30] Schechter, M.: *Spectra of Partial Differential Operators*. Amsterdam: North Holland, 1986.

[31] Schmidt, G.: On scattering by time dependent perturbation. Indiana Univ. Math. J. **24**, 925–935 (1974).

[32] Simon, B.: *Quantum Mechanics for Hamiltonians Defined as Quadratic Forms*. Princeton: Princeton University Press, 1971.

[33] Sylvester, J., Uhlman, G.: A global uniqueness theorem for an inverse boundary value problem. Ann. of Math. **125**, 153–169 (1987).

[34] Uhlmannn, G., Vasy, A.: Fixed energy inverse problem for exponentially decreasing potentials. Methods Appl. Anal. **9**, 239–247 (2002).

[35] Weder, R.: *Spectral and Scattering Theory for Wave Propagation in Perturbed Stratified Media*. Applied Mathematical Sciences **87**, New York: Springer, 1991.

[36] Weder, R.: Generalized limiting absorption method and multidimensional inverse scattering theory. Math. Methods Appl. Sci. **14**, 509–524 (1991).

[37] Weder, R.: Global uniqueness at fixed energy in multidimensional inverse scattering theory. Inverse Problems **7**, 927–938 (1991).

[38] Weder, R.: Inverse scattering for N–Body systems with time–dependent potentials. In: Chavent, G., Sabatier, P.C. (eds.), *Inverse Problems of Wave Propagation and Diffraction*, Lecture Notes in Physics **486**, Berlin: Springer, 1997, pp 27–46 .

[39] Weder, R.: Multidimensional inverse problems in perturbed stratified media. J. Differential Equations **152**, 191–239 (1999).

[40] Yafaev, D.R.: Conditions for the unitarity of wave operators under scattering by a nonstationary potential. Funktional Anal. i Prilozhen **14** , 91–92 (1980) [engl. transl. Funct. Anal. Appl. **14**, 325–326 (1980)].

[41] Yafaev, D.R.: The virial theorem and conditions for wave operators to be unitary in scattering by a nonstationary potential. Trudy Mat. Inst. Steklov **159**, 210–217 (1983).

[42] Yafaev, D.R.: On resonant scattering for time–periodic perturbations. In *Journées Equations aux Derivées Partielles (Saint Jean de Monts, 1990)*. Palaiseau: Ecole Polytech., 1990, Exp. No. 4, 8 pp.

[43] Yajima, K.: Scattering theory for Schrödinger equations with potentials periodic in time. J. Math. Soc. Japan **29**, 729–743 (1977).

[44] Yajima, K., Kitada H.: Bound states and scattering states for time periodic Hamiltonians. Ann. Inst. H. Poincaré Phys. Théor. XXXIX, 145–157 (1983).

[45] Yajima, K.: Existence of solutions for Schrödinger evolution equations. Commun. Math. Phys. **40**, 415–426 (1987).

[46] Yajima, K.: Exponential decay of quasi–stationary states of time–periodic Schrödinger equation with short range potential. Sci. Papers College Arts Sci. Univ. Tokyo **40**, 27–36 (1990).

[47] Yajima, K.: Schrödinger evolution equations with magnetic fields. J. Anal. Math. **56**, 29–76 (1991).

[48] Yokoyama, K.: Mourre theory for time–periodic systems, Nagoya Math. J. **149**, 193–210 (1998).